Elusive extremal graphs

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Abstract

We study the uniqueness of optimal solutions to extremal graph theory problems. Lovász conjectured that every finite feasible set of subgraph density constraints can be extended further by a finite set of density constraints so that the resulting set is satisfied by an asymptotically unique graph. This statement is often referred to as saying that "every extremal graph theory problem has a finitely forcible optimum". We present a counterexample to the conjecture. Our techniques also extend to a more general setting involving other types of constraints.

1 Introduction

Many problems in extremal graph theory do not have asymptotically unique solutions. As an example, consider the problem of minimizing the sum of the induced subgraph densities of K_3 and its complement. It can be shown that this sum is minimized by any n-vertex graph where all vertices have degrees equal to $n/2$. For example, the complete bipartite graph $K_{n/2,n/2}$, the union of two $(n/2)$-vertex complete graphs, or (with high probability) an Erdős-Rényi random graph

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$G_{n,1/2}$ all minimize the sum. However, the structure of an optimal solution can be made unique by adding additional density constraints. In our example, setting the triangle density to be zero forces the structure to be that of the complete bipartite graph with parts of equal sizes. Alternatively, fixing the density of cycles of length four forces the structure to be that of a quasirandom graph. The most frequently quoted conjecture concerning dense graph limits is a conjecture of Lovász [30–32, 34], stated as Conjecture 1 below, which asserts that this is a general phenomenon for a large class of problems in extremal graph theory. We disprove this conjecture.

We treat Conjecture 1 in the language of the theory of graph limits, to which we provide a brief introduction. This theory has offered analytic tools to represent and analyze large graphs, and has led to new tools and views on various problems in mathematics and computer science. We refer the reader to a monograph by Lovász [32] for a detailed introduction to the theory. The theory is also closely related to the flag algebra method of Razborov [45], which changed the landscape of extremal combinatorics [48] by providing solutions and substantial progress on many long-standing open problems, see, e.g. [2–6, 23–25, 28, 29, 37, 38, 45–47].

The *density* of a k-vertex graph H in G, denoted by $d(H, G)$, is the probability that a uniformly randomly chosen k-tuple of vertices of G induce a subgraph isomorphic to H; if G has less than k vertices, we set $d(H, G) = 0$. A sequence of graphs $(G_n)_{n\in\mathbb{N}}$ is *convergent* if the sequence $(d(H, G_n))_{n\in\mathbb{N}}$ is convergent for every graph H. In this paper, we only consider convergent sequences of graphs where the number of vertices tends to infinity.

A convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is *finitely forcible* if there exist graphs $H_1, \dots, H_\ell$ with the following property: if $(G'_n)_{n\in\mathbb{N}}$ is another convergent sequence of graphs such that

$$\lim_{n\to\infty} d(H_i, G_n) = \lim_{n\to\infty} d(H_i, G'_n)$$

for every $i = 1, \dots, \ell$, then

$$\lim_{n\to\infty} d(H, G_n) = \lim_{n\to\infty} d(H, G'_n)$$

for every graph H. For example, one of the classical results on quasirandom graphs [13, 50, 51] is equivalent to saying that a sequence of Erdős-Rényi random graphs is finitely forcible (by densities of 4-vertex subgraphs) with probability one. Lovász and Sós [33] generalized this result to graph limits corresponding to stochastic block models (which are represented by step graphons). Additional examples of finitely forcible sequences can be found, e.g., in [19, 34].

One of the most commonly cited problems concerning graph limits is the following conjecture of Lovász, which is often referred to as saying that "every extremal problem has a finitely forcible optimum", see [32, p. 308]. The conjecture appears in various forms, also sometimes as a question, in the literature; we include only some of the many references with its statement below.

Conjecture 1 (Lovász [30, Conjecture 3], [31, Conjecture 9.12], [32, Conjecture 16.45], and [34, Conjecture 7]). *Let $H_1, \dots, H_\ell$ be graphs and $d_1, \dots, d_\ell$ reals. If there exists a convergent sequence of graphs with the limit density of H_i equal to d_i, $i = 1, \dots, \ell$, then there exists a finitely forcible such sequence.*

Our main result (Theorem 18) implies that the conjecture is false. The proof of Theorem 18 uses analytic objects (graphons) provided by the theory of graph limits, however, unlike in earlier results concerning finitely forcible graph limits, we have parametrized a whole family of objects that represent large graphs and applied analytic tools to understand the behavior of these objects. To control the objects in the family, we have adopted the method of decorated constraints. This method builds on the flag algebra method of Razborov, which we have mentioned earlier, and was used to enforce an asymptotically unique structure of graphs. In the setting of the proof of Theorem 18, the method is used to enforce a unique structure inside a large graph while keeping some of its other parts variable.

We now explain the general strategy of the proof of Theorem 18 in more detail. By fixing finitely many subgraph densities, we will enforce the asymptotic structure of graphs except for certain specific parts. The structure of the variable parts will depend on a vector $z \in [0,1]^{\mathbb{N}}$ in a controlled way, in particular, one of the variable parts of the graphs will encode the values z_i as edge densities. We will also have a countable set of polynomial inequalities that the values z_i will satisfy for any graph in the family that we consider; this will be enforced by the variable parts and the subgraph densities. This set of polynomial inequalities is constructed iteratively, and each time we add new constraints, the graphs in the family change in a controlled way. We eventually construct a large subset of $[0,1]^{\mathbb{N}}$, which is defined by a countable set of polynomial inequalities, such that *any* subgraph density in the graph corresponding to z is determined by finitely many coordinates of z. This will yield a counterexample to Conjecture 1.

While we present our arguments to avoid forcing the unique structure of a graph by subgraph densities, our approach also applies to additional graph parameters, as discussed in Section 8. For example, one may hope that even if it is not possible to force an asymptotically unique structure by fixing a finite number of subgraph densities, it may be possible to force a unique typical structure, i.e., obtain an asymptotically unique structure maximizing the entropy. However, this hope is dismissed by Theorem 28.

The proof of Theorem 18 presented in this paper is not constructive. However, as we discuss in Section 8, a constructive proof can be obtained using the main result of [15], which is an earlier (but unrefereed) version of [16]. It is also worth noting that all graphs $H_1, \dots, H_\ell$ in the statement of Theorem 18 have at most 225 vertices; we give further details in Section 8 after presenting the proof of Theorem 18.

The paper is organized as follows. In Section 2, we start by introducing the notation that we use. We next give a broad outline and set up some basic concepts for the proof of Theorem 18 in Section 3, including a summary of the construction of a family of graphons with finitely forcibly structure, which is parameterized by $z \in [0,1]^{\mathbb{N}}$. We also state the properties of the construction that we need for the proof. We next detail the proof, assuming such a construction exists. We first present an iterative way of restricting values of well-behaved functions in countably many variables in Section 4. In Section 5, we show how this implies Theorem 18. In Section 6, we detail the construction of the family of graphons and prove that it can be forced by finitely many constraints. Finally, in Section 7, we analyze the dependence of the graphons in the family on $z \in [0,1]^{\mathbb{N}}$.

2 Preliminaries

In this section, we introduce notation used throughout the paper. We start with some general notation. The set of all positive integers is denoted by $\mathbb{N}$, the set of all non-negative integers by $\mathbb{N}_0$, and the set of integers between 1 and k (inclusive) by $[k]$. All measures considered in this paper are Borel measures on $\mathbb{R}^d$, $d \in \mathbb{N}$. If a set $X \subseteq \mathbb{R}^d$ is measurable, then we write $|X|$ for its measure, and if X and Y are two measurable sets, then we write $X \sqsubseteq Y$ if $|X \setminus Y| = 0$.

We will be working with vectors with both finitely many and countably infinitely many coordinates, and sometimes with double coordinates. If $X \subseteq [0,1]$ and J is a set, then the set X^J is the set of vectors with coordinates from X indexed by J. In particular, $[0,1]^{\mathbb{N}}$ denotes the set of all vectors with coordinates indexed by positive integers and each coordinate from the interval $[0,1]$. We will always understand X^d to be $X^{[d]}$. If $x \in \mathbb{R}^J$, $J' \subseteq J$ and $\varepsilon > 0$, then we define

$$N_{\varepsilon,J'}(x) = \{x' \text{ s.t. } |x_j - x'_j| < \varepsilon \text{ for } j \in J', \text{ and } x_j = x'_j \text{ for } j \notin J'\}.$$

If $X \subseteq \mathbb{R}^J$, $J' \subseteq J$ and $\varepsilon > 0$, then we set

$$N_{\varepsilon,J'}(X) = \bigcup_{x \in X} N_{\varepsilon,J'}(x).$$

If $J' = J$, we will omit the second index in the subscript. Finally, we use $\overline{N}_{\varepsilon,J'}(X)$ for the closure of $N_{\varepsilon,J'}(X)$.

We will often work with the following set of double indexed vectors:

$$\mathbb{A} = \prod_{i \in \mathbb{N}} [0,1]^{i+1}.$$

The elements of $\mathbb{A}$ are vectors, each having coordinates indexed by $\mathbb{N}$ and the i-th coordinate being a vector from $[0,1]^{i+1}$. For example, if $\vec{a} \in \mathbb{A}$, then $\vec{a}_2 \in [0,1]^3$ and the coordinates of $\vec{a}_2$ are $\vec{a}_{2,1}$, $\vec{a}_{2,2}$ and $\vec{a}_{2,3}$. We define the following bijection between the elements of $[0,1]^{\mathbb{N}}$ and $\mathbb{A}$: if $a \in [0,1]^{\mathbb{N}}$, we define $\vec{a} \in \mathbb{A}$ so that $a = (\vec{a}_{1,1}, \vec{a}_{1,2}, \vec{a}_{2,1}, \vec{a}_{2,2}, \vec{a}_{2,3}, \ldots)$. Using this bijection $a \to \vec{a}$, we will understand the corresponding elements of $[0,1]^{\mathbb{N}}$ and $\mathbb{A}$ to be the same, and we use the arrow overscript to distinguish between the corresponding elements of $[0,1]^{\mathbb{N}}$ and $\mathbb{A}$, i.e., whether the vector is single or double indexed.

We say that the function $f : [0,1]^{\mathbb{N}} \to \mathbb{R}$ is *totally analytic* if the following two properties hold:

- for any finite multiset M of positive integers, the partial derivative
$$\frac{\partial^{|M|}}{(\partial z)^M} f(z)$$
is a continuous function with respect to the product topology on $[0,1]^{\mathbb{N}}$, and
- for any fixed $(z^0_{n+1}, z^0_{n+2}, \ldots) \in [0,1]^{\mathbb{N}\setminus[n]}$, the function
$$f(z_1, z_2, \ldots, z_n, z^0_{n+1}, \ldots)$$
can be extended to an analytic function of the variables $(z_1, z_2, \ldots, z_n)$ on an open set containing $[0,1]^n$.

We say that two totally analytic functions are *ε-close* if the functions and all their first partial derivatives differ by at most ε on $[0,1]^{\mathbb{N}}$. We say that a function f is a *polynomial* in z_i, $i \in \mathbb{N}$, if it is a polynomial in finitely many of the variables z_i, $i \in \mathbb{N}$. Clearly, every polynomial in z_i, $i \in \mathbb{N}$, is totally analytic.

2.1 Graph limits

We now introduce some notions from the theory of graph limits, which we have not covered in Section 1. To simplify our notation, if G is a graph, we will write $|G|$ for its order, i.e., its number of vertices.

Each convergent sequence of graphs $(G_n)_{n\in\mathbb{N}}$ can be associated with an analytic limit object, which is called a graphon. Formally, a *graphon* is a symmetric measurable function W from $[0,1)^2$ to the unit interval $[0,1]$, where *symmetric* refers to the property that $W(x,y) = W(y,x)$ for all $x, y \in [0,1)$. We remark that it is more usual to define graphons as symmetric measurable functions from the closed unit square $[0,1]^2$; however, both definitions represent the same notion and it is more convenient for us to work with half-open intervals. Graphons are often visualized by a unit square filled in with different shades of gray representing the values of $W(x,y)$ (with 0 being represented by white and 1 by black). We will often refer to the points of $[0,1)$ as *vertices*, and we think of a graphon W intuitively as a continuous version of an adjacency matrix. Because of this, the origin $(0,0)$ is always drawn in the top left corner in visualizations of graphons.

We next present a connection between convergent sequences of graphs and graphons. Given a graphon W, a *W-random graph* of order n is a graph obtained from W by sampling n points from $[0,1)$ independently and uniformly at random, associating each point with one of the n vertices, and joining two vertices by an edge with probability $W(x,y)$ where x and y are the points in $[0,1)$ associated to them. The density of a graph H in a graphon W, denoted by $d(H,W)$, is the probability that a W-random graph of order $|H|$ is isomorphic to H. Observe that the expected density of H in a W-random graph of order $n \geq |H|$ is equal to $d(H,W)$. It also holds that the density of H in a W-random graph is concentrated around its expected density by standard martingale arguments. We will also use the labeled version of the density of a graph H in a graphon W, which is the probability that a W-random graph is the graph H and the i-th vertex of the W-random graph is the i-th vertex of the graph H. This probability will be denoted by $\tau(H,W)$ and it holds that

$$\tau(H,W) = \int_{[0,1)^{V(H)}} \prod_{\{u,v\}\in E(H)} W(x_u,x_v) \prod_{\{u,v\}\in\binom{V(H)}{2}\setminus E(H)} (1-W(x_u,x_v))dx_{V(H)}.$$

Note that $d(H,W)$ is equal to $\tau(H,W)$ multiplied by $|H|!/|\mathrm{Aut}(H)|$.

A graphon W is a *limit* of a convergent sequence of graphs $(G_n)_{n\in\mathbb{N}}$ if

$$\lim_{n\to\infty} d(H,G_n) = d(H,W)$$

for every graph H. A standard martingale argument yields that if W is a graphon, then a sequence of W-random graphs with increasing orders converges to W with probability one. In the other direction, Lovász and Szegedy [35] showed that every convergent sequence of graphs has a limit graphon. However, the limit is in general not unique. We say that two graphons W_1 and W_2 are *weakly isomorphic* if $d(H,W_1) = d(H,W_2)$ for every graph H, i.e., the graphons W_1 and W_2 are limits of the same convergent sequences of graphs. Borgs, Chayes and Lovász [8] proved that two graphons W_1 and W_2 are weakly isomorphic if and only if there exists a third graphon W and measure preserving maps $\varphi_1,\varphi_2 : [0,1) \to [0,1)$ such that $W_i(\varphi_i(x),\varphi_i(y)) = W(x,y)$ for almost every $(x,y)\in[0,1)^2$ and $i=1,2$.

Because of Conjecture 1, we are interested in graphons that are uniquely determined, up to weak isomorphism, by the densities of a finite set of graphs. Formally, a graphon W is *finitely forcible* if there exist graphs $H_1,\dots,H_\ell$ such that if a graphon W' satisfies $d(H_i,W') = d(H_i,W)$ for $i\in[\ell]$, then W and W' are weakly isomorphic. The following characterization of finitely forcible graphons is one of the motivations coming from extremal graph theory for Conjecture 1.

Proposition 1. *A graphon W is finitely forcible if and only if there exist graphs $H_1,\dots,H_\ell$ and reals $\alpha_1,\dots,\alpha_\ell$ such that*

$$\sum_{i=1}^{\ell} \alpha_i d(H_i,W) \leq \sum_{i=1}^{\ell} \alpha_i d(H_i,W')$$

for every graphon W', with equality if only if W and W' are weakly isomorphic.

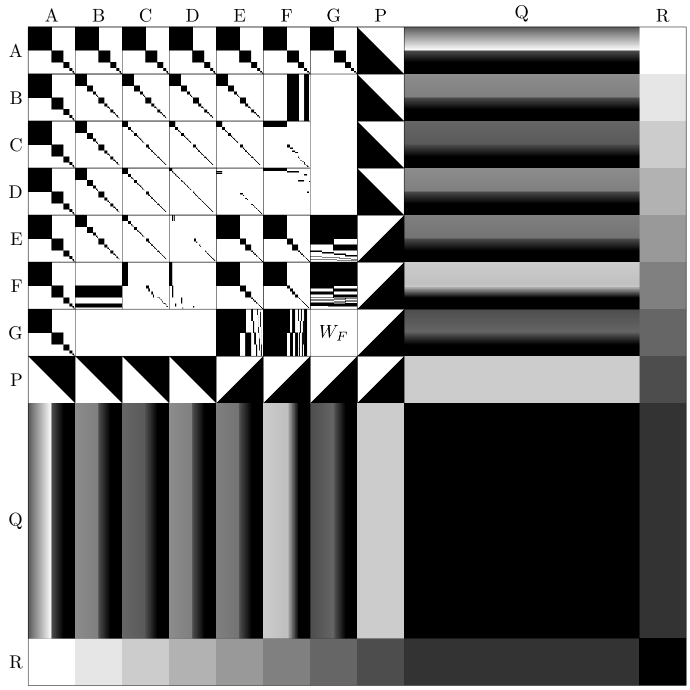

Figure 1: The graphon W_0 from Theorem 2.

While the initial results on the structure of finitely forcible graphons suggested that all finitely forcible graphons could possess a simple structure [34, Conjectures 9 and 10], this turned out not to be the case [14, 20, 21] in general. In particular, every graphon can be a subgraphon of a finitely forcible graphon [16]. Here, we say that W' is a *subgraphon* of W if there exists a non-null subset $Z \subseteq [0,1)$ and a measure preserving map $\varphi : Z \to [0, |Z|)$ such that $W(x,y) = W'(\varphi(x)/|Z|, \varphi(y)/|Z|)$ for all $x, y \in Z$.

Theorem 2 (Cooper et al. [16])**.** *There exist graphs $H_1, \ldots, H_\ell$ with the following property. For every graphon W_F, there exist a graphon W_0 and reals $d_1, \ldots, d_\ell \in [0,1]$ such that any graphon W with $d(H_i, W) = d_i$ for all $i \in [\ell]$ is weakly isomorphic to W_0, and W_F is a subgraphon of W_0 formed by a $1/14$ fraction of its vertices. In particular, the graphon W_0 is finitely forcible.*

The graphon W_0 from Theorem 2 is visualized in Figure 1. We will revisit the proof of Theorem 2 in Subsection 6.3 and additional properties of the graphons W_0, which we state in the next proposition, will be needed in Section 7. A *dyadic square* is a square of the form $[(i-1)/2^k, i/2^k) \times [(j-1)/2^k, j/2^k)$, where $k \in \mathbb{N}_0$ and $i, j \in [2^k]$. Recall that the *standard binary representation* of a real number is the binary representation such that set of digits equal to zero is not finite.

Proposition 3. *For every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ with the following property. Let W_F and W'_F be any two graphons such that the densities of their dyadic* *squares of sizes at least 2^{-k} agree up to the first k bits after the decimal point in the standard binary representation, and let W_0 and W_0' be the corresponding graphons from Theorem 2 that contain W_F and W_F', respectively. The L_1-distance between the graphons W_0 and W_0' viewed as functions in $L^1([0,1)^2)$ is at most ε.*

We finish by presenting graph limit analogues of several standard graph theory notions. Let W be a graphon. The *degree* of a vertex $x \in [0,1)$ is defined as

$$\deg_W(x) = \int_{[0,1)} W(x,y)\,\mathrm{d}y\,.$$

When it is clear from the context which graphon we are referring to, we will omit the subscript, i.e., we will just write $\deg(x)$ instead of $\deg_W(x)$. The *neighborhood* of x is the set of all y such that $W(x,y) > 0$. This set is denoted by $N_W(x)$, where we again drop the subscript if it is clear from the context. Note that $\deg_W(x) \le |N_W(x)|$ and the inequality can be strict. If A is a non-null subset of $[0,1)$, then the *relative degree* of a vertex $x \in [0,1)$ with respect to A is

$$\deg_W^A(x) = \frac{\int_A W(x,y)\,\mathrm{d}y}{|A|},$$

i.e., the degree of x with respect to A normalized by the measure of A. Similarly, $N_W^A(x) = N_W(x) \cap A$ is the *relative neighborhood* of x with respect to A. As in the previous cases, we drop the subscripts if W is clear from the context.

2.2 Decorated constraints

The arguments related to forcing the structure of graphons introduced in Section 6 are based on the method of decorated constraints from [20,21]. This method uses the language of the flag algebra method of Razborov developed in [45], which has had many significant applications in extremal combinatorics [48].

We now give a formal definition of decorated constraints. A *density expression* is a polynomial in graphs with real coefficients. Formally, it can be defined recursively as follows. A real number or a graph are density expressions, and if D_1 and D_2 are density expressions, then $D_1 + D_2$ and $D_1 \cdot D_2$ are also density expressions. An *ordinary density constraint* is an equality between two density expressions. The value of a density expression for a graphon W is obtained by substituting $\tau(H,W)$ for each graph H. A graphon W *satisfies* an ordinary density constraint if the density expressions on the two sides have the same value for W. Note that we obtain an equivalent notion if we use $d(H,W)$ instead of $\tau(H,W)$ when computing the value of a density of expression (adjusting the coefficients appropriately).

We next introduce a formally stronger type of density constraint: a rooted density constraint. A *rooted graph* is a graph H with m distinguished vertices, which are labeled with the elements of $[m]$ and are referred to as the *roots*. Let H_0 be the subgraph of H induced by the roots. We then define $\tau(H,W|x_1,\ldots,x_m)$ to be the probability that the W-random graph is H, conditioned on x_i being associated with the i-th root. It follows that

$$\tau(H,W|x_1,\ldots,x_m)=\int\limits_{[0,1)^U}\prod_{\{u,v\}\in E(H)}W(x_u,x_v)\prod_{\{u,v\}\in\binom{V(H)}{2}\setminus E(H)}(1-W(x_u,x_v))\,\mathrm{d}x_U,$$

where U is the set of the non-root vertices of H. Two rooted graphs are *compatible* if the subgraphs induced by their roots are isomorphic through an isomorphism preserving their labels. A *rooted density expression* is a density expression containing compatible rooted graphs. We define $\tau(C,W|x_1,\ldots,x_m)$ to be the value of the expression C when each rooted graph H is substituted with $\tau(H,W|x_1,\ldots,x_m)$. Let $C_1=C_2$ be a constraint where all rooted graphs are compatible. We say that the graphon W *satisfies* the constraint $C_1=C_2$ if

$$\tau(C_1,W|x_1,\ldots,x_m)=\tau(C_2,W|x_1,\ldots,x_m)$$

holds for almost every m-tuple $x_1,\ldots,x_m\in[0,1)$. Note that if the m-tuple $x_1,x_2,\ldots,x_m$ is chosen in such a way that a W-random graph with the vertices associated with $x_1,\ldots,x_m$ is H_0 with probability zero, then both sides of the above equality are equal to zero.

A graphon W is said to be *partitioned* if there exist an integer $k\in\mathbb{N}$, positive reals $a_1,\ldots,a_k$ summing to one, and distinct reals $d_1,\ldots,d_k\in[0,1]$, such that for each $i\in[k]$, the set of vertices in W with degree d_i has measure a_i. The set of all vertices with degree d_i will be referred to as a *part*; the *size* of a part is its measure and its *degree* is the common degree of its vertices. For example, the graphon depicted in Figure 1 has ten parts and all but Q have the same size. If X and Y are parts, then we will refer to the restriction of W to $X\times Y$ as to the *tile* $X\times Y$. The following lemma was proven in [20, 21].

Lemma 4. *Let $a_1,\ldots,a_k$ be positive real numbers such that $a_1+\ldots+a_k=1$ and let $d_1,\ldots,d_k\in[0,1]$ be distinct reals. There exists a finite set of ordinary density constraints $\mathcal{C}$ such that a graphon satisfies all constraints in $\mathcal{C}$ if and only if it is a partitioned graphon with parts of sizes $a_1,\ldots,a_k$ and degrees $d_1,\ldots,d_k$.*

We next introduce a formally even stronger type of density constraint. Fix $a_1,\ldots,a_k$ and $d_1,\ldots,d_k$ as in Lemma 4. A *decorated* graph is a graph H with $m\le|H|$ distinguished vertices, which are called *roots* and labeled with $[m]$, and with each vertex (distinguished or not) assigned one of the k parts, which will be called the *decoration* of a vertex. We allow $m=0$, i.e., a decorated graph can have no roots. Suppose that $x_1,\ldots,x_m\in[0,1)$ are given such that each x_i belongs to the part that the i-th root is decorated with. We then define $\tau(H,W|x_1,\ldots,x_m)$ to be the probability that a W-random graph is H, conditioned on the i-th root being associated with x_i, and conditioned on each non-root vertex being associated with a point from the part that it is decorated with. Two decorated graphs are *compatible* if the subgraphs induced by their roots are isomorphic through an isomorphism preserving the labels of the roots and the decorations of all vertices. A *decorated density expression* C is an expression that contains compatible decorated graphs only, and we define $\tau(C, W|x_1, \ldots, x_m)$ in the obvious way. A *decorated density constraint* is an equality between two density expressions that contain compatible decorated graphs instead of ordinary graphs; we will often say *decorated constraint* instead of decorated density constraint.

Before defining when a decorated constraint is satisfied, let us introduce a way of visualizing decorated graphs and decorated constraints, which was also used in [14, 16, 20]. We will draw decorated graphs as graphs with each vertex labeled by its decoration; the root vertices will be drawn as squares and non-root vertices as circles. The decorated constraint will be presented as an expression containing decorated graphs, where the roots of all the graphs appearing in the constraint will be presented in the same position in all the graphs of the constraint; this establishes the correspondence between the roots in individual decorated graphs. A solid line between two vertices will represent an edge and a dashed line a non-edge. The absence of a line between two root vertices indicates that the decorated constraint should hold for both the root graph containing this edge and the one not containing it. The absence of a line between a non-root vertex and another vertex represents the sum over all decorated graphs with and without this edge. In particular, if there are k such lines missing, the drawing represents the sum of 2^k decorated graphs.

We next define when a graphon W satisfies a decorated constraint; the definition is followed by an example of evaluating a decorated graph. Suppose that W is a partitioned graphon and $C_1 = C_2$ is a decorated constraint such that each decorated graph in the constraint has (the same) m roots. The graphon W *satisfies* the constraint C if

$$\tau(C_1, W|x_1, \ldots, x_m) = \tau(C_2, W|x_1, \ldots, x_m)$$

for almost every m-tuple $x_1, \ldots, x_m \in [0, 1)$ such that x_i belongs to the part that the i-th root is decorated with.

To illustrate this definition, we give an example. Consider the partitioned graphon W that has two parts A and B, each of size $1/2$, and that is equal to $2/3$, $1/3$ and 1 on the tiles $A \times A$, $A \times B$ and $B \times B$, respectively. The graphon is depicted in Figure 2; the three decorated graphs depicted in Figure 2 are evaluated to $8/27$, $4/9$ and $16/243$ (from left to right) for any admissible choices of the roots.

In [21], it was shown that the expressive power of ordinary and decorated density constraints is the same. To be precise, the following was proven.

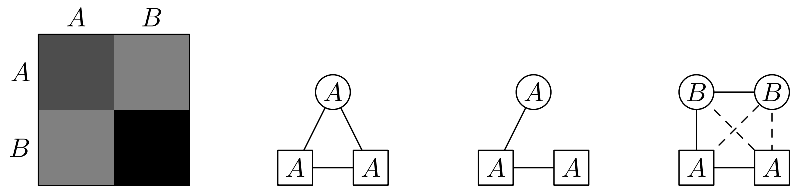

Figure 2: An example of evaluating decorated constraints. The depicted partitioned graphon has two parts A and B, each of size $1/2$, and is equal to $2/3$, $1/3$ and 1 on the tiles $A \times A$, $A \times B$ and $B \times B$, respectively. The three decorated graphs that are depicted are evaluated to $8/27$, $4/9$ and $16/243$ (from left to right) for any admissible choices of the roots.

Lemma 5. *Let $k \in \mathbb{N}$, let $a_1, \ldots, a_k$ be positive real numbers summing to one, and let $d_1, \ldots, d_k$ be distinct reals between zero and one. For every decorated density constraint C, there exists an ordinary density constraint C' such that every partitioned graphon W with parts of sizes $a_1, \ldots, a_k$ and degrees $d_1, \ldots, d_k$ satisfies C if and only if it satisfies C'.*

Lemmas 4 and 5 were used in [14, 16, 20, 21] to argue that certain partitioned graphons are finitely forcible in the following way. Suppose that W is the unique partitioned graphon (up to weak isomorphism) that satisfies a finite set $\mathcal{C}$ of decorated constraints. By Lemmas 4 and 5, the graphon W is the unique graphon that satisfies a finite set $\mathcal{C}'$ of ordinary decorated constraints. It follows that the graphon W is the unique graphon such that the density of each subgraph H appearing in a constraint of $\mathcal{C}'$ is equal to $d(H, W)$. In particular, W is finitely forcible. In this paper, we follow a similar line of arguments to show that a certain family of graphons can be forced by finitely many subgraph densities.

We next state two auxiliary lemmas, which can be found explicitly stated in [14, Lemmas 7 and 8]. The first lemma says that if a graphon W_0 is finitely forcible, then its structure may be forced on a part of a partitioned graphon. The second lemma is implicit in [34, proof of Lemma 2.7 or Lemma 3.3].

Lemma 6. *Let k be an integer, $a_1, \ldots, a_k$ positive reals summing to one, and $d_1, \ldots, d_k$ distinct reals between zero and one. For every finitely forcible graphon W_0 and every $m \in [k]$, then there exists a finite set $\mathcal{C}$ of decorated constraints such that any partitioned graphon W with parts $A_1, \ldots, A_k$ of sizes $a_1, \ldots, a_k$ and degrees $d_1, \ldots, d_k$ satisfies $\mathcal{C}$ if and only if the tile $A_m \times A_m$ is weakly isomorphic to W_0, i.e., there exist measure preserving maps $\varphi_0 : [0, 1) \to [0, 1)$ and $\varphi_m : [0, a_m) \to A_m$ such that $W(\varphi_m(x a_m), \varphi_m(y a_m)) = W_0(\varphi_0(x), \varphi_0(y))$ for almost every $(x, y) \in [0, 1)^2$.*

Lemma 7. *Let $F : X \times Z \to [0, 1]$ be a measurable function, $X, Z \subseteq [0, 1)$, such* *that*

$$\int_Z F(x,z)F(x',z)\,dz = \xi$$

for almost every $(x,x') \in X^2$*. Then, it holds that*

$$\int_Z F(x,z)^2\,dz = \xi$$

for almost every $x \in X$*.*

3 General setup

The proof of our main theorem, Theorem 18, builds on techniques developed in relation to the finite forcibility of graphons. We find a finite set of density constraints with the property that any graphon that satisfies these constraints has most of its structure forced, except for a small part that can vary in a controlled manner. In particular, we will construct a family of graphons with structure depending on a bounding sequence, which is a notion that we define below, and the graphons in the family are indexed by elements of a subset of $[0,1]^{\mathbb{N}}$.

Before proceeding further, we need to introduce some definitions. We first define a linear order $\preceq$ on finite multisets of natural numbers. Each such multiset A can be associated with a vector $\chi(A) \in \mathbb{N}_0^{\mathbb{N}}$, where $\chi(A)_n$ is the multiplicity of the containment of n in A. Let $\Sigma(A)$ be the sum of the elements of A, i.e.,

$$\Sigma(A) = \sum_{n\in\mathbb{N}} n \cdot \chi(A)_n\,.$$

If A and B are two multisets of natural numbers, then we will say that $A \preceq B$ if either $A = B$, or $\Sigma(A) < \Sigma(B)$, or $\Sigma(A) = \Sigma(B)$ and the first entry different in $\chi(A)$ and $\chi(B)$ is smaller in $\chi(A)$. Let M_i be the i-th multiset in this linear order, i.e., $M_1 = \emptyset$, $M_2 = \{1\}$, $M_3 = \{2\}$, $M_4 = \{1,1\}$, etc. Since there exists at least one multiset M with $\Sigma(M) = n$ for each n, we get the following observation.

Observation 8. *It holds that* $\Sigma(M_i) \le i$ *for every* $i \in \mathbb{N}$*.*

We call a sequence $P = (p_i, l_i, u_i)_{i\in\mathbb{N}}$ a *bounding sequence* if

- each p_i is a polynomial and $p_i(z) \in [0,1]$ for all $z \in [0,1]^{\mathbb{N}}$,
- the coefficient of $\pi_{i,j}$ in the polynomial p_i at the monomial z^{M_j} satisfies $|\pi_{i,j}| \le 2^{-2^j}/9$,
- all the partial derivatives of p_i have values between -1 and $+1$ (inclusive) on $[0,1]^{\mathbb{N}}$,
- $l_i \in [0,1]$, $u_i \in [0,1]$ and $l_i \le u_i$.

Each bounding sequence describes a subset of $[0,1]^{\mathbb{N}}$, which simply contains all $z \in [0,1]^{\mathbb{N}}$ such that $p_i(z) \in [l_i, u_i]$ for each i, and we will use such subsets to describe a subfamily graphons. The second condition in the definition of a bounding sequence is needed, in particular, in order to make sure we can encode the polynomial in the graphon. We say that a bounding sequence P' is a *strengthening* of the bounding sequence P if they agree on all triples except some triples where $(p_i, l_i, u_i) = (0, 0, 1)$ in P. Note that if P' is a strengthening of P, then the subset of $[0,1]^{\mathbb{N}}$ associated with P' is contained in the subset associated with P. If P' is a strengthening of P that agrees with P on the first k elements, then we say that P' is a *k-strengthening*.

In Section 6, we will construct, for any bounding sequence P, a family $\mathcal{W}_P$ of graphons $W_P(z)$. Each of the graphons $W_P(z)$ is described by a bounding sequence $P = (p_i, l_i, u_i)_{i\in\mathbb{N}}$ and a vector $z \in [0,1]^{\mathbb{N}}$ that satisfies $p_i(z) \in [l_i, u_i]$ for each i. These families of graphons will have the following properties.

Theorem 9. *There exists a finite set of graphs $H_1, H_2, \ldots, H_\ell$ and an integer D such that the following holds. For any bounding sequence $P = (p_i, l_i, u_i)_{i\in\mathbb{N}}$, there exists a polynomial q of degree at most D in ℓ variables such that the following two statements are equivalent for any graphon W:*

- *The graphon W is weakly isomorphic to a graphon $W_P(z)$ for some $z \in [0,1]^{\mathbb{N}}$ such that $p_i(z) \in [l_i, u_i]$ for all $i \in \mathbb{N}$.*
- *The graphon W satisfies $q(d(H_1, W), d(H_2, W), \ldots, d(H_\ell, W)) = 0$.*

Furthermore, if $W_P(z)$ and $W_P(z')$ are weakly isomorphic, then $z = z'$.

We defer the precise description of the family $\mathcal{W}_P$ and the proof of Theorem 9 to Section 6. For the proof of our main result, we will need additional properties of graphons in the family $\mathcal{W}_P$, which we summarize in the next two lemmas. In Section 7, we define a function $t_{P,H} : [0,1]^{\mathbb{N}} \to \mathbb{R}$ for each bounding sequence P and each graph H. The next proposition asserts that $t_{P,H}$ is a totally analytic function and it describes the density of H for those z that satisfy P.

Lemma 10. *For any finite graph H and bounding sequence $P = (p_i, l_i, u_i)_{i\in\mathbb{N}}$, the function $t_{P,H}(z)$ is totally analytic. Furthermore, for every $z \in [0,1]^{\mathbb{N}}$ satisfying $p_i(z) \in [l_i, u_i]$ for all $i \in \mathbb{N}$, it holds that $t_{P,H}(z) = \tau(H, W_P(z))$.*

We will also need the following proposition, which roughly says that if we change P in a controlled way, then the functions $t_{P,H}$ do not change by much.

Lemma 11. *For any finite graph H, bounding sequence P, and $\varepsilon > 0$, there exists an integer k such that for any k-strengthening P' of P, the function $t_{P',H}$ is ε-close to $t_{P,H}$.*

Lemmas 10 and 11 will be proven in Section 7.

4 Stabilization

Our main argument involves a sequence of steps, each resulting in a strengthening of a bounding sequence that defines a family of graphons as above. The whole process is carried in a careful way so that it is still possible to vary some of the parameters without changing the densities of any given finite set of graphs. In this section, we present analytic lemmas needed to perform these steps.

We first prove the following analytic statement. The proof follows the lines of the proof of Grönwall's inequality [22], see also [7], however, we present an entire proof here for completeness.

Lemma 12. *For every integer $n \in \mathbb{N}$, reals $L > 0$, $D > 0$ and $\varepsilon > 0$, and closed interval $[a, b] \subseteq \mathbb{R}$, there exist $\delta > 0$ and $\varepsilon_0 \in (0, \varepsilon)$ such that the following holds. Let $g : [a, b] \to \mathbb{R}^n$ be a function defined on $[a, b]$ and let $V = \{(x, N_\varepsilon(g(x))) : x \in [a, b]\}$. Furthermore, let $f : V \to \mathbb{R}^n$ be an analytic function such that $f(x, g(x)) = 0$ for all $x \in [a, b]$, f and each partial derivative of f is L-Lipschitz on V, and the absolute value of the determinant of the Jacobi matrix*

$$\left(\frac{\partial}{\partial y_i} f_j(x, y)\right)_{i,j=1}^{n}$$

is at least D on V. For every analytic function $\widehat{f} : V \to \mathbb{R}^n$ that is δ-close to f and for every $(x_0, y_0) \in V$ such that $x_0 \in [a, b]$, $\widehat{f}(x_0, y_0) = 0$ and $\|g(x_0) - y_0\|_\infty \le \varepsilon_0$, there exists a function $\widehat{g} : [a, b] \to V$ such that $\widehat{g}(x_0) = y_0$, $\|\widehat{g}(x) - g(x)\|_\infty \le \varepsilon$ and $\widehat{f}(x, \widehat{g}(x)) = 0$ for all $x \in [a, b]$.

Proof. Fix a point $(x_0, y_0) \in V$ and the functions f, $\widehat{f}$ and g for the proof. Let $A(x, y_1, \ldots, y_n)$ be the $n \times n$ Jacobi matrix of f with respect to $y_1, y_2, \ldots, y_n$, and let $h(x, y_1, \ldots, y_n)$ be the partial derivative of f with respect to x. We define $\widehat{A}$ and $\widehat{h}$ for $\widehat{f}$ in an analogous way. Take δ small enough so that the absolute value of the determinant of $\widehat{A}$ is at least $D/2$ on the whole set V (note that this choice of δ depends only on n, D and L); in particular, $\widehat{A}$ is invertible on whole set V. By the Implicit Function Theorem, if $\widehat{g}$ is defined for $x \in (a, b)$, $\widehat{f}(x, \widehat{g}(x)) = 0$ and $\widehat{A}(x, g(x))$ is invertible, then $\widehat{g}$ can be extended to an open neighborhood of x. Let $U \subseteq [a, b]$ be the maximal interval containing x_0 that $\widehat{g}$ can be extended to in a way that $\widehat{f}(x, \widehat{g}(x)) = 0$ and $\|\widehat{g}(x) - g(x)\|_\infty \le \varepsilon$.

We next analyze the derivative of $f(x, g(x))$ as a function of x on (a, b):

$$\frac{\partial}{\partial x} f(x, g(x)) = h(x, g(x)) + A(x, g(x)) \frac{\partial}{\partial x} g(x).$$

Since this derivative is zero on (a, b), it follows that

$$\frac{\partial}{\partial x} g(x) = -(A(x, g(x)))^{-1} h(x, g(x)).$$

Similarly, it holds that

$$\frac{\partial}{\partial x}\widehat{g}(x) = -(\widehat{A}(x, \widehat{g}(x)))^{-1}\widehat{h}(x, \widehat{g}(x))$$

for any x in the interior of U. Let $m(x, y) = -(A(x, y))^{-1}h(x, y)$ and analogously $\widehat{m}(x, y) = -(\widehat{A}(x, y))^{-1}\widehat{h}(x, y)$. Observe that both m and $\widehat{m}$ are analytic and $\|m(x, y) - \widehat{m}(x, y)\|_2 \leq \rho$ for all $(x, y) \in V$, where ρ is a constant depending only on n, D, L, ε and δ. In addition, if n, D, L, and ε are fixed, ρ can be made arbitrarily close to zero by choosing δ sufficiently small.

Let us now consider the function

$$G(x) = \|g(x) - \widehat{g}(x)\|_2^2 = \sum_{i=1}^{n}(g_i(x) - \widehat{g}_i(x))^2$$

defined on U. Note that $G(x_0) \leq n\varepsilon_0^2$. We next analyze the derivative of $G(x)$.

$$\begin{aligned}
\frac{\partial}{\partial x}G(x) &= 2\left\langle \frac{\partial}{\partial x}g(x) - \frac{\partial}{\partial x}\widehat{g}(x), g(x) - \widehat{g}(x)\right\rangle \\
&= 2\left\langle m(x, g(x)) - \widehat{m}(x, \widehat{g}(x)), g(x) - \widehat{g}(x)\right\rangle \\
&= 2\left\langle m(x, g(x)) - m(x, \widehat{g}(x)), g(x) - \widehat{g}(x)\right\rangle \\
&+ 2\left\langle m(x, \widehat{g}(x)) - \widehat{m}(x, \widehat{g}(x)), g(x) - \widehat{g}(x)\right\rangle .
\end{aligned}$$

It follows that

$$\left|\frac{\partial}{\partial x}G(x)\right| \leq K \cdot G(x) + 2\rho\sqrt{G(x)} \leq (K + \rho)G(x) + \rho, \tag{1}$$

where K depends only on n, L, D and δ (since each entry of the matrix $A(x, y)^{-1}$ is at most $n!L^n/D$, and each entry of the matrix is Lipschitz on V for a Lipschitz constant depending only on n, L and D). We now consider the function $H : U \to \mathbb{R}$ that is defined as

$$H(x) = \left(G(x) + \frac{\rho}{K + \rho}\right)e^{-(x-x_0)(K+\rho)}. \tag{2}$$

The derivative of H is equal to

$$\frac{\partial}{\partial x}H(x) = \left(\frac{\partial}{\partial x}G(x) - (K + \rho)G(x) - \rho\right)e^{-(x-x_0)(K+\rho)} \leq 0,$$

where the inequality follows from (1). This implies that $H(x) \leq H(x_0)$ for $x \in U$ such that $x \geq x_0$. Hence, we obtain using (2) that

$$\left(G(x) + \frac{\rho}{K + \rho}\right)e^{-(x-x_0)(K+\rho)} \leq G(x_0) + \frac{\rho}{K + \rho} \leq n\varepsilon_0^2 + \frac{\rho}{K + \rho}.$$

In particular, it is possible to choose ε_0 and δ small enough (the choice of δ makes ρ small enough) that it holds that

$$\left(n\varepsilon_0^2 + \frac{\rho}{K+\rho}\right) e^{(b-x_0)(K+\rho)} < \varepsilon^2.$$

Hence, we obtain that

$$G(x) \le \left(n\varepsilon_0^2 + \frac{\rho}{K+\rho}\right) e^{(x-x_0)(K+\rho)} - \frac{\rho}{K+\rho} < \varepsilon^2.$$

for all $x \in U$ such that $x \ge x_0$.

We next show that the interval $[x_0, b]$ is contained in U. Let β be the supremum of U and suppose that $\beta < b$. Since the left limit $y_\beta = \lim_{x \nearrow \beta} \widehat{g}(x)$ satisfies that $\widehat{f}(\beta, y_\beta) = 0$, we can assume that $\beta \in U$. However, this implies that $\|\widehat{g}(\beta) - g(\beta)\|_\infty \le \|\widehat{g}(\beta) - g(\beta)\|_2 < \varepsilon$ and the function $\widehat{g}$ can be extended to a neighborhood of β, contradicting the fact that β is the supremum of U.

An analogous argument using the function

$$H(x) = \left(G(x) + \frac{\rho}{K+\rho}\right) e^{(x-x_0)(K+\rho)}$$

yields that it is possible to choose ε_0 and δ small enough that $G(x) \le \varepsilon^2$ for all $x \in U$ such that $x \le x_0$, which then yields that the interval $[a, x_0]$ is a subset of U. This completes the proof. □

To state the next lemma, we need a definition. A *variable system* consists of

- a sequence of non-degenerate closed intervals $U_k \subseteq [0,1]$, $k \in \mathbb{N}$, and
- a sequence of closed sets V_k, $k \in \mathbb{N}$, with $V_1 \subseteq U_1$ and $V_k \subseteq V_{k-1} \times U_k$.

We say that a variable system is *c-strong* for $c > 0$ if the measure of the set $\{x' : (x, x') \in V_k\}$ is at least $c|U_k|$ for every $x \in V_{k-1}$. A variable system is *k_0-finite* if $V_k = V_{k-1} \times U_k$ for each $k > k_0$.

We will need the following lemma on variable systems.

Lemma 13. *Suppose that $(U_k, V_k)_{k\in\mathbb{N}}$ is a c-strong variable system with $c > 0$, and T is a non-zero analytic function on V_ℓ, $\ell \in \mathbb{N}$. For any $c' < c$, there exist subsets $V'_k \subseteq V_k$, $k \in [\ell]$, such that $(U_k, V'_k)_{k\in\mathbb{N}}$, with $V'_k = V_k \cap (V'_{k-1} \times U_k)$ for $k > \ell$, is a c'-strong variable system and T is non-zero on V_ℓ everywhere.*

Proof. We first consider the case when $\ell = 1$. Let Z be the set of all $x \in U_1$ such that $T(x) = 0$. Since T is analytic and U_1 is bounded, the set Z is finite. Let X be the union of open δ-neighborhoods of the points in Z, where $\delta > 0$ is sufficiently small so that $|X| < (c - c')|U_1|$. The set $V'_1 = V_1 \setminus X$ is a closed set that satisfies the statement of the lemma.

The argument for $\ell > 1$ extends the one presented for $\ell = 1$ but is more complex. Let $\varepsilon = c - c'$. Let Z be the set of all $x \in V_\ell$ such that

$$T(x_1, \ldots, x_\ell) = 0 .$$

Since T is analytic and not everywhere zero, Z is a closed set with measure zero.

Let $X \subseteq V_\ell$ be the set of points with $|T(x)| < \delta$ where $\delta > 0$ is sufficiently small so that the measure of X is less than $\varepsilon^\ell \prod_{j=1}^{\ell} |U_j|$. Note that X is open. For every $i \in [\ell-1]$, let X_i be the set of the points $x \in V_i$ such that the measure of the set $\{x' \in \prod_{i<j\le\ell} U_j, (x,x') \in X\}$ is greater than $\varepsilon^{\ell-i} \prod_{j=i+1}^{\ell} |U_j|$. Let $X_\ell = X$. Observe that each of the sets X_i, $i \in [\ell]$, is open. In addition, the measure of X_1 is at most $\varepsilon|U_1|$, and for any $i \in [\ell-1]$ and $x \in V_i$, if the measure of the points $x' \in U_i$ such that $(x,x') \in X_{i+1}$ is greater than $\varepsilon|U_{i+1}|$, then $x \in X_i$.

We now define V_i' for each $i \in [\ell]$ in increasing order by setting $V_i' = (V_i \setminus X_i) \cap \big(V_{i-1}' \times U_i\big)$. Since each set X_i is open, we get that V_i' is closed. If $x \in V_i \setminus X_i$ for $i \in [\ell-1]$, then the measure of $x' \in U_{i+1}$ such that $(x,x') \in X_{i+1}$ is at most $\varepsilon|U_{i+1}|$. It follows that for every $x \in V_i'$, the measure of $x' \in U_{i+1}$ such that $(x,x') \in V_{i+1}'$ is at least $(c-\varepsilon)|U_{i+1}| = c'|U_{i+1}|$. Since the measure of X_1 is at most $\varepsilon|U_1|$, it also follows that the measure of V_1' is at least $(c-\varepsilon)|U_1| = c'|U_1|$. We set $V_k' = V_k \cap (V_{k-1}' \times U_k)$ for $k > \ell$, and conclude that $(U_k, V_k')_{k\in\mathbb{N}}$is a c'-strong variable system. □

A *stabilizing system* $S = (U_k, V_k, I_k, J_k, d_k, b_k, w_k)_{k\in\mathbb{N}}$ is a sequence of 7-tuples such that

- $(U_k, V_k)_{k\in\mathbb{N}}$ is a variable system,
- I_k is a subset of $[k]$,
- J_k is a subset of $[k+1]$ with $|I_k| = |J_k|$,
- $d_k \in [k+1]$ such that $d_k \notin J_k$,
- $b_k \in (0,1)^{k+1}$ and b_{k,d_k} is in the interior of U_k, and
- w_k is an analytic function from $V_{k-1} \times U_k \to [0,1]^{J_k \cup \{d_k\}}$ such that $w_k(x)_{d_k} = x_k$ for every $x \in V_{k-1} \times U_k$ and $w_k(x, b_{k,d_k})_i = b_{k,i}$ for every $x \in V_{k-1}$ and $i \in J_k$.

The functions w_k, $k \in \mathbb{N}$, will be referred to as *stabilizers*. We will say that a stabilizing system is *c-strong* if the variable system $(U_k, V_k)_{k\in\mathbb{N}}$ is c-strong. Finally, a stabilizing system is *k_0-finite* if the variable system $(U_k, V_k)_{k\in\mathbb{N}}$ is k_0-finite, and $J_k = \emptyset$ and $d_k = 1$ for every $k > k_0$; we will say that a stabilizing system is *finite* if it is k_0-finite for some $k_0 \in \mathbb{N}$.

Given a stabilizing system S, we define the *extended stabilizers* $\widetilde{w}_k : V_{k-1} \times U_k \to [0,1]^{k+1}$ by

$$\widetilde{w}_k(x_1, x_2, \ldots, x_k)_i = \begin{cases} w_k(x_1, x_2, \ldots, x_k)_i & \text{if } i \in J_k \cup \{d_k\}, \text{ and} \\ b_{k,i} & \text{otherwise.} \end{cases}$$

We also define a function $\widetilde{w}_{\leq k} : V_{k-1} \times U_k \to [0,1]^{2+3+\ldots+(k+1)}$ as

$$\widetilde{w}_{\leq k}(x_1, \ldots, x_k) = (\widetilde{w}_1(x_1), \widetilde{w}_2(x_1, x_2), \ldots, \widetilde{w}_k(x_1, x_2, \ldots, x_k)).$$

Finally, we write $b_{>k}$ for the point $(b_{k+1}, b_{k+2}, \ldots)$.

Suppose that S is a stabilizing system and $t_k : [0,1]^{\mathbb{N}} \to [0,1]$, $k \in \mathbb{N}$, are totally analytic functions. Let $M_k(\vec{a})$ for $k \in \mathbb{N}$ and $\vec{a} \in \mathbb{A}$ be the matrix

$$M_k(\vec{a}) = \left(\frac{\partial}{\partial \vec{a}_{k,j}} t_i(a) \right)_{i \in [k], j \in [k+1]}.$$

We say that the system S *stabilizes* the functions t_k, $k \in \mathbb{N}$, if for every $k \in \mathbb{N}$ and $x \in V_{k-1} \times U_k$, it holds that

(P1) the matrix $M_k(\widetilde{w}_{\leq k}(x), b_{>k})$, restricted to the entries indexed by $I_k \times J_k$ is invertible, and

(P2) for each $i \in I_k$, and $x \in V_k$, we have

$$t_i(\widetilde{w}_{\leq k}(x), b_{>k}) = t_i(\widetilde{w}_{\leq k-1}(x_1, \ldots, x_{k-1}), b_k, b_{>k}).$$

Note that the properties (P1) and (P2) always hold if $I_k = J_k = \emptyset$.

We are now ready to state the following lemma.

Lemma 14. *Let $S = (U_k, V_k, I_k, J_k, d_k, b_k, w_k)_{k \in \mathbb{N}}$ be a finite stabilizing system that stabilizes totally analytic functions t_k, $k \in \mathbb{N}$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Suppose that for each $k \in \mathbb{N}$, we have $b'_k \in (0,1)^{k+1}$ such that b'_{k,d_k} is in the interior of U_k and $\|b_k - b'_k\|_\infty \leq \delta$, and we have $t'_k : [0,1]^{\mathbb{N}} \to [0,1]$ totally analytic functions such that t'_k is δ-close to t_k. Then there exist functions $w'_k : V_{k-1} \times U_k \to (0,1)^{J_k \cup \{d_k\}}$ such that the stabilizing system $S' = (U_k, V_k, I_k, J_k, d_k, b'_k, w'_k)_{k \in \mathbb{N}}$ stabilizes t'_k, $k \in \mathbb{N}$, and $\|w_k(x) - w'_k(x)\|_\infty \leq \varepsilon$ for every $k \in \mathbb{N}$ and $x \in V_{k-1} \times U_k$.*

Proof. Let k_0 be such that S is k_0-finite. We construct the stabilizers inductively for $k \in [k_0]$. Suppose that we have constructed $w'_1, w'_2, \ldots, w'_{k-1}$ such that for each $i \in [k-1]$ and $x \in V_{i-1} \times U_i$, $\|w'_i(x) - w''_i(x)\|_\infty < \varepsilon'$, where the value of $\varepsilon' < \varepsilon$ will be chosen later. For $(x, y) \in U_k \times [0,1]^{J_k}$, let $a(x,y) \in [0,1]^{k+1}$ be the vector with $a(x,y)_i = y_i$ for $i \in J_k$, $a(x,y)_{d_k} = x$, and $a(x,y)_i = b_{k,i}$ otherwise. Define

$a'(x,y)$ analogously using $b'_{k,i}$. Since V_{k-1} is compact, there exists a constant L such that for every $\widetilde{x} \in V_{k-1}$ the function $f_{\widetilde{x}} : U_k \times [0,1]^{J_k} \to [0,1]^{I_k}$, defined as

$$f_{\widetilde{x}}(x,y) = (t_i(\widetilde{w}_{\leq k-1}(\widetilde{x}), a(x,y), b_{>k}) - t_i(\widetilde{w}_{\leq k-1}(\widetilde{x}), b_k, b_{>k}))_{i \in I_k}$$

has the property that the function and each partial derivative is L-Lipschitz. Similarly, define the function $\widehat{f_{\widetilde{x}}} : U_k \times (0,1)^{J_k} \to [0,1]^{I_k}$, $\widetilde{x} \in V_{k-1}$ to be

$$\widehat{f_{\widetilde{x}}}(x,y) = \left(t'_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), a'(x,y), b'_{>k}) - t'_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), b'_k, b'_{>k})\right)_{i \in I_k}.$$

We claim that for any δ', we can take δ and ε' small enough so that for each $\widetilde{x} \in V_{k-1}$, $\widehat{f_{\widetilde{x}}}$ is δ'-close to $f_{\widetilde{x}}$. Fix $i \in [k]$. Since t_i is continuous on the product topology on $[0,1]^{\mathbb{N}}$, it is uniformly continuous with respect to the metric $d(x,y) = \sum_{n \in \mathbb{N}} 2^{-n}|x_n - y_n|$. In particular, this means that there exists $\varepsilon'' > 0$ so that if $||x-y||_\infty < \varepsilon''$, $x, y \in [0,1]^{\mathbb{N}}$, then

$$|t_i(x) - t_i(y)| < \delta'/4. \tag{3}$$

We next obtain the following estimate.

$$\begin{aligned}
&\left|t'_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), a'(x,y), b'_{>k}) - t_i(\widetilde{w}_{\leq k-1}(\widetilde{x}), a(x,y), b_{>k})\right| \\
\leq &\left|t'_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), a'(x,y), b'_{>k}) - t_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), a'(x,y), b'_{>k})\right| \\
+ &\left|t_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), a'(x,y), b'_{>k}) - t_i(\widetilde{w}_{\leq k-1}(\widetilde{x}), a(x,y), b_{>k})\right| \leq \delta + \delta'/4.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left|t'_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), b'_k, b'_{>k}) - t_i(\widetilde{w}_{\leq k-1}(\widetilde{x}), b_k, b_{>k})\right| \\
\leq &\left|t'_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), b'_k, b'_{>k}) - t_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), b'_k, b'_{>k})\right| \\
+ &\left|t_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), b'_k, b'_{>k}) - t_i(\widetilde{w}_{\leq k-1}(\widetilde{x}), b_k, b_{>k})\right| \leq \delta + \delta'/4.
\end{aligned}$$

Hence, if $\varepsilon' < \varepsilon''$ and $\delta < \delta'/4$, we obtain that

$$\begin{aligned}
\left|f_{\widetilde{x}}(x,y)_i - \widehat{f_{\widetilde{x}}}(x,y)_i\right| \leq &\left|t'_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), a'(x,y), b'_{>k}) - t_i(\widetilde{w}_{\leq k-1}(\widetilde{x}), a(x,y), b_{>k})\right| \\
&+ \left|t'_i(\widetilde{w}'_{\leq k-1}(\widetilde{x}), b'_k, b'_{>k}) - t_i(\widetilde{w}_{\leq k-1}(\widetilde{x}), b_k, b_{>k})\right| \leq \delta'.
\end{aligned}$$

An analogous argument applies to each partial derivative of t_i.

Let $T_k(\widetilde{x}, z)$, $\widetilde{x} \in V_{k-1}$ and $z \in [0,1]^{k+1}$, be the determinant of the submatrix of $M_k(\widetilde{w}_{\leq k-1}(\widetilde{x}), z, b_{>k})$ formed by the entries indexed by $I_k \times J_k$. Since the set $V_{k-1} \times \overline{N}_{\varepsilon, J_k}(\widetilde{w}_k(V_{k-1} \times U_k))$ is closed for every $\varepsilon > 0$, we can decrease $\varepsilon > 0$ (note that this strengthens the conclusion of the lemma) so that T_k^2 has a positive minimum D^2 on the set $V_{k-1} \times \overline{N}_{\varepsilon, J_k}(\widetilde{w}_k(V_{k-1} \times U_k))$, i.e., $|T_k|$ is at least D on this set, for a positive real D. We apply Lemma 12 with $n = |J_k|$, L, D, ε as above, and $[a,b] = U_k$ to obtain δ' and ε_0. We then make sure that δ and ε' are small enough so that $\widehat{f_{\widetilde{x}}}$ is δ'-close to $f_{\widetilde{x}}$ for every $\widetilde{x} \in V_{k-1}$.

Using the triangle inequality, we get the following estimate:

$$\begin{aligned}\|b'_k - \widetilde{w}_k(\widetilde{x}, b'_{k,d_k})\|_\infty \le & \|b'_k - b_k\|_\infty + \|b_k - \widetilde{w}_k(\widetilde{x}, b_{k,d_k})\|_\infty \\ & + \|\widetilde{w}_k(\widetilde{x}, b_{k,d_k}) - \widetilde{w}_k(\widetilde{x}, b'_{k,d_k})\|_\infty.\end{aligned}$$

The first summand on the right hand side is at most δ, the middle one is zero and the last summand can be made arbitrarily small by choosing δ small enough (δ that is universal for all choices of $\widetilde{x} \in V_{k-1}$ exists since V_{k-1} is compact). Hence, by choosing δ small enough, we can guarantee that $\|b'_k - w_k(b'_{k,d_k})\|_\infty \le \varepsilon_0$. Since $f_{\widetilde{x}}(w_k(x)) = 0$ for every $x \in U_k$, Lemma 12 implies that there exists $g_{\widetilde{x}} : U_k \to [0,1]^{J_k \cup \{d_k\}}$ such that $\widehat{f}_{\widetilde{x}}(g_{\widetilde{x}}(x)) = 0$ and $\|w_k(\widetilde{x}, x) - g_{\widetilde{x}}(x)\| \le \varepsilon$ for all $x \in U_k$. We set $w'_k(\widetilde{x}, x) = g_{\widetilde{x}}(x)$ for $(\widetilde{x}, x) \in V_{k-1} \times U_k$. □

We next consider stabilizing systems with a stronger property. We say that a stabilizing system $(U_k, V_k, I_k, J_k, d_k, b_k, w_k)_{k\in\mathbb{N}}$ that stabilizes totally analytic functions t_k, $k \in \mathbb{N}$, is *m-excellent* if the rank of the matrix $M_k(\widetilde{w}_{\le k}(x), b_{>k})$ is equal to $|J_k|$ for all $x \in V_{k-1} \times U_k$ and $k \in [m]$. Note that the definition of stabilizing implies that the submatrix of $M_k(\widetilde{w}_{\le k}(x), b_{>k})$ formed by the entries indexed by $I_k \times J_k$ has rank $|J_k|$, so we require that this submatrix has the same rank as the whole matrix $M_k(\widetilde{w}_{\le k}(x), b_{>k})$.

We next prove two auxiliary lemmas on excellent stabilizing systems.

Lemma 15. *Let S be an m-excellent stabilizing system that stabilizes totally analytic functions t_k, $k \in \mathbb{N}$. It holds that*

$$t_\ell(\widetilde{w}_{\le k}(x), b_{>k}) = t_\ell(\widetilde{w}_{\le k-1}(x_1, \ldots, x_{k-1}), b_k, b_{>k})$$

for all $x \in V_{k-1} \times U_k$, $\ell \in [k]$ and $k \in [m]$.

Proof. Fix integers $k \in [m]$ and $\ell \in [k]$. Since the rank of matrix $M_k(\widetilde{w}_{\le k}(x), b_{>k})$ is equal to $|J_k|$ for every $x \in V_{k-1} \times U_k$, there exist functions $c_i : V_{k-1} \times U_k \to \mathbb{R}$, $i \in I_k$, such that

$$\frac{\partial}{\partial \vec{a}_{k,j}} t_\ell(\widetilde{w}_{\le k}(x), b_{>k}) = \sum_{i \in I_k} c_i(x) \frac{\partial}{\partial \vec{a}_{k,j}} t_i(\widetilde{w}_{\le k}(x), b_{>k})$$

for every $j \in [k+1]$. Let $F_i(x) = t_i(\widetilde{w}_{\le k}(x), b_{>k})$, $i \in [k]$. We now analyze the derivative of the function $F_\ell(x)$:

$$\begin{aligned}\frac{\partial}{\partial x_k} F_\ell(x) &= \sum_{j=1}^{k+1} \frac{\partial}{\partial \vec{a}_{k,j}} t_\ell(\widetilde{w}_{\le k}(x), b_{>k}) \frac{\partial}{\partial x_k} (\widetilde{w}_{\le k}(x))_j \\ &= \sum_{i \in I_k} \sum_{j=1}^{k+1} c_i(x) \frac{\partial}{\partial \vec{a}_{k,j}} t_i(\widetilde{w}_{\le k}(x), b_{>k}) \frac{\partial}{\partial x_k} (\widetilde{w}_{\le k}(x))_j\end{aligned}$$

$$= \sum_{i \in I_k} c_i(x) \frac{\partial}{\partial x_k} F_i(x).$$

Since the system S stabilizes all functions t_i with $i \in I_k$, it follows that $\frac{\partial}{\partial x_k} F_i(x) = 0$ for all $x \in V_{k-1} \times U_k$, which implies that $\frac{\partial}{\partial x_k} F_\ell(x) = 0$. We conclude that

$$t_\ell(\widetilde{w}_{\leq k}(x), b_{>k}) = t_\ell(\widetilde{w}_{\leq k-1}(x_1, \ldots, x_{k-1}), b_k, b_{>k})$$

for all $x \in V_{k-1} \times U_k$. $\square$

Lemma 16. *Let $S = (U_k, V_k, I_k, J_k, d_k, b_k, w_k)_{k \in \mathbb{N}}$ be a c-strong stabilizing system that stabilizes totally analytic functions t_k, $k \in \mathbb{N}$, and that is $(m-1)$-excellent but not m-excellent. For any $c' < c$ and $\varepsilon > 0$, there exists a c'-strong m-finite stabilizing system $S' = (U'_k, V'_k, I'_k, J'_k, d'_k, b'_k, w'_k)_{k \in \mathbb{N}}$ such that $b_k = b'_k$ for $k \neq m$, $U'_k = U_k$, $I'_k = I_k$, $J'_k = J_k$ and $d'_k = d_k$ for $k \in [m-1]$, $\|w_k(x) - w'_k(x)\|_\infty \leq \varepsilon$ for any $x \in V'_{k-1} \times U'_k$, $k \in [m-1]$, $I_m \subsetneq I'_m$, $J_m \subsetneq J'_m$, $\|b_m - b'_m\|_\infty \leq \varepsilon$, and $U'_k \subseteq [b'_{k,d'_k} - \varepsilon, b'_{k,d'_k} + \varepsilon]$ for $k \geq m$. In addition, it also holds that $||\widetilde{w}'_m(x) - b'_m||_\infty \leq \varepsilon$ for every $x \in V'_{m-1} \times U'_m$.*

Proof. Recall that $M_m(\vec{a})$ is the $m \times (m+1)$ Jacobian matrix of the partial derivatives of the functions $t_1, t_2, \ldots, t_m$ with respect to the variables $\vec{a}_{m,i}$, $i \in [m+1]$. Let r be the maximum rank of $M_m(\vec{a})$ taken over all $\vec{a} = (\widetilde{w}_{\leq m}(x), b_{>m})$, $x \in V_{m-1} \times U_m$. Since the stabilizing system is not m-excellent, we have that $r > |J_m|$. Let $x \in V_{m-1} \times U_m$ be a point where the rank $M_m(\vec{a})$ is equal to r; by the analyticity of the functions $t_1, \ldots, t_m$, we can choose x such that x_m is arbitrarily close to b_{m,d_m}. Set $b'_m = w_m(x_m)$ and let $w'_1, \ldots, w'_{m-1}$ be the stabilizers for the system S with b_m replaced with b'_m. By Lemma 14, if b'_m is sufficiently close to b_m (which can be arranged by choosing x_m sufficiently close to b_{m,d_m}), the stabilizers $w'_1, \ldots, w'_{m-1}$ exist and it holds that $\|w_k(x) - w'_k(x)\|_\infty \leq \varepsilon$ for any $x \in V_{k-1} \times U_k$, $k \in [m-1]$. Moreover, if ε is sufficiently small, the rank of the matrix $M_m(\widetilde{w}'_{\leq m-1}(x_1, \ldots, x_{m-1}), b'_m, b_{>m})$ stays at least r and the rank of its submatrix formed by the entries indexed by $I_m \times J_m$ stays $|J_m|$.

Let $I'_m \supseteq I_m$ and $J'_m \supseteq J_m$ be subsets of $[m]$ and $[m+1]$, respectively, such that the rank of the submatrix of $M_m(\widetilde{w}'_{\leq m-1}(x_1, \ldots, x_{m-1}), b'_m, b_{>m})$ formed by the entries indexed by $I'_m \times J'_m$ is r. Choose $d'_m \in [m+1] \setminus J'_m$ arbitrarily and let $T_m(x)$ be the determinant of the submatrix of $M_m(\widetilde{w}'_{\leq m-1}(x), b'_m, b_{>m})$, $x \in V_{m-1}$. Set $U'_k = U_k$ for $k \in [m-1]$, $U'_m = [b'_{m,d'_m} - \varepsilon, b'_{m,d'_m} + \varepsilon] \cap [0,1]$, and $U'_k = [b_{k,1} - \varepsilon, b_{k,1} + \varepsilon] \cap [0,1]$, otherwise. By Lemma 13, there exist closed subsets $V'_k \subseteq V_k$, $k \in [m-1]$, such that the variable system $(U'_k, V'_k)_{k \in \mathbb{N}}$ with $V'_k = V'_{k-1} \times U'_k$ for $k \geq m$ is c'-strong and $T_m(x) \neq 0$ for all $x \in V'_{m-1}$. By the Implicit Function Theorem and the compactness of V'_{k-1}, there exists a non-trivial closed interval $U \subseteq U'_m$ containing b'_{m,d'_m} such that a stabilizer w'_m exists on $V'_{m-1} \times U$ and $||\widetilde{w}'_m(x) - b'_m||_\infty \leq \varepsilon$ for every $x \in V'_{m-1} \times U$. We replace U'_m with U and set $I'_k = I_k$, $J'_k = J_k$ and $d'_k = d_m$ for $k \in [m-1]$. We also set $I'_k = \emptyset$, $J'_k = \emptyset$,

$d'_k = 1$ and $w'_k(z) := z$ for $k > m$, and $b'_k = b_k$ for $k \neq m$. We have obtained an m-finite stabilizing system $S' = (U'_k, V'_k, I'_k, J'_k, d'_k, b'_k, w'_k)_{k\in\mathbb{N}}$ with $J'_k = J_k$ for $k \in [m-1]$ and $J_m \subsetneq J'_m$, completing the proof. $\square$

We next define a partial order on sequences of finite subsets of integers. Given two such sequences $\mathcal{J} = (J_k)_{k\in\mathbb{N}}$ and $\mathcal{J}' = (J'_k)_{k\in\mathbb{N}}$, we say that $\mathcal{J} \prec \mathcal{J}'$ if there exists an integer m such that

- $J_k = J'_k$ for $k < m$, and
- $J_m \subset J'_m$, i.e., J_m is a proper subset of J'_m.

In other words, we consider the partial order defined by the lexicographic order together with the partial order given by subset containment. We will also write $\mathcal{J} \preceq \mathcal{J}'$ if $\mathcal{J} \prec \mathcal{J}'$ or $\mathcal{J} = \mathcal{J}'$.

We are now ready to prove the main lemma of this section.

Lemma 17. *Suppose that $S = (U_k, V_k, I_k, J_k, d_k, b_k, w_k)_{k\in\mathbb{N}}$ is a c-strong stabilizing system that stabilizes totally analytic functions t_k, $k \in \mathbb{N}$. Let $\varepsilon > 0$, $c' \in (0, c)$ and $m \in \mathbb{N}$. There exists a c'-strong m-excellent stabilizing system $S' = (U'_k, V'_k, I'_k, J'_k, d'_k, b'_k, w'_k)_{k\in\mathbb{N}}$ that stabilizes the functions t_k, $k \in \mathbb{N}$, and that satisfies $(J_k)_{k\in\mathbb{N}} \preceq (J'_k)_{k\in\mathbb{N}}$ and $\|b_k - b'_k\|_\infty < \varepsilon$ for all $k \in \mathbb{N}$.*

Moreover, the following holds. Let k_0 be the largest integer such that $J'_k = J_k$ for all $k \in [k_0]$. For every $k \in [k_0]$, it holds that $I'_k = I_k$, $d'_k = d_k$, $U'_k = U_k$, $V'_k \subseteq V_k$, and $\|w_k(x) - w'_k(x)\|_\infty < \varepsilon$ for any $x \in V'_{k-1} \times U'_k$. It also holds that $||\widetilde{w}'_k(x) - b'_k||_\infty < \varepsilon$ for every $x \in V'_{k-1} \times U'_k$ and $k > k_0$.

Proof. Set $L = (m+1)!$, $\varepsilon' = \varepsilon/(L+1)$ and $\delta = (c - c')/(L+1)$. We keep successively applying Lemma 16. At each step, we choose the smallest $m' \leq m$ such that the current stabilizing system is not m'-excellent and apply Lemma 16 with ε' and $c - i\delta$ in the i-th step. Note that in each step the sequence $(J_k)_{k\in\mathbb{N}}$ increases in the order defined by $\prec$ and this increase is witnessed by J_k with $k \leq m' \leq m$. It is not difficult to see that there can be at most L increases in total. Hence, the whole procedure terminates after at most L steps with an m-excellent stabilizing system $S' = (U'_k, V'_k, I'_k, J'_k, d'_k, b_k, w_k)_{k\in\mathbb{N}}$. By the choice of δ, the resulting system is c'-strong, and the choice of ε' implies that $\|b_k - b'_k\|_\infty < \varepsilon$ for all $k \in \mathbb{N}$. Moreover, if the first k_0 elements of the sequence $(J_k)_{k\in\mathbb{N}}$ stayed the same, i.e., $J'_k = J_k$ for all $k \in [k_0]$, then we only applied Lemma 16 with $m' > k_0$, therefore, $U'_{k_0} = U_{k_0}$, $V'_{k_0} \subseteq V_{k_0}$, and $\|w_{k_0}(x) - w'_{k_0}(x)\|_\infty \leq L\varepsilon' < \varepsilon$ for any $x \in V'_{k_0-1} \times U'_{k_0}$. $\square$

5 Main result

We are now ready to prove the main theorem.

Theorem 18. *There exists a family of graphons $\mathcal{W}$, graphs $H_1, \ldots, H_\ell$ and reals $d_1, \ldots, d_\ell$ such that*

- *a graphon W is weakly isomorphic to a graphon contained in $\mathcal{W}$ if and only if $d(H_i, W) = d_i$ for every $i \in [\ell]$, and*
- *no graphon in $\mathcal{W}$ is finitely forcible, i.e., for all graphs $H'_1, \ldots, H'_r$ and reals $d'_1, \ldots, d'_r$, the family $\mathcal{W}$ contains either zero or infinitely many graphons W with $d(H'_i, W) = d'_i$, $i \in [r]$.*

Proof. The family $\mathcal{W}$ will be a family of graphons $\mathcal{W}_P$, which we introduced in Section 3 and we formally define in Section 6, for a suitable choice of bounding sequence P. In this proof, we will only use Theorem 9, Lemma 10 and Lemma 11, which can be found in Section 3, and the results in Section 4.

Fix an enumeration G_k, $k \in \mathbb{N}$, of all graphs. We will iteratively apply the results of Section 4 to construct bounding sequences P^m, $m \in \mathbb{N}$, which converge to the sought bounding sequence P, and corresponding finite stabilizing systems $S^m = (U^m_k, V^m_k, I^m_k, J^m_k, d^m_k, b^m_k, w^m_k)_{k\in\mathbb{N}}$, which stabilize the densities of G_k. Initially, we set P^0 to be the sequence where each element is $(0, 0, 1)$ and S^0 to be the 0-finite stabilizing system with $b^0_k = (1/2, \ldots, 1/2)$, i.e., $U^0_k = [0, 1]$, $V^0_k = [0, 1]^k$, $I^0_k = J^0_k = \emptyset$, $d^0_k = 1$ and $w^0_k(x) = x$ for all $k \in \mathbb{N}$.

Let t^m_k be the function t_{P^m,G_k} introduced at the end of Section 3, let $Q^m_{k,\varepsilon} = \overline{N}_\varepsilon(\widetilde{w}_{\le k}(V^m_{k-1} \times U^m_k)) \subseteq [0, 1]^{2+3+\ldots+(k+1)}$ for $k \in \mathbb{N}$, and let δ_m be the value of δ from Lemma 14 applied with S^{m-1}, t^{m-1}_k, and $\varepsilon = 1/2^{m+1}$. The bounding sequence P^m and the stabilizing system S^m will satisfy the following properties for every $m \in \mathbb{N}$.

- The bounding sequence P^m is a strengthening of P^{m-1} and contains an infinite number of elements equal to $(0, 0, 1)$.
- Each function t^m_k, $k \in \mathbb{N}$, is δ_m-close to t^{m-1}_k.
- The stabilizing system S^m stabilizes the functions t^{m-1}_k, $k \in \mathbb{N}$, and S^m is m-finite, $(1/2 + 1/2^{m+1})$-strong and m-excellent.
- $\|b^m_k - b^{m-1}_{k-1}\|_\infty < 1/2^{m+1}$ for every $k \in \mathbb{N}$.
- $(J^{m-1}_k)_{k\in\mathbb{N}} \preceq (J^m_k)_{k\in\mathbb{N}}$, and if k_0 is the largest integer such that $J^m_k = J^{m-1}_k$ for all $k \in [k_0]$, then $I^m_k = I^{m-1}_k$, $d^m_k = d^{m-1}_k$, $U^m_k = U^{m-1}_k$, $V^m_k \subseteq V^{m-1}_k$, and $\|w^m_k(x) - w^{m-1}_k(x)\|_\infty < 1/2^m$ for any $x \in V^m_{k-1} \times U^m_k$ with $k \in [k_0]$.
- $Q^m_{m,1/2^m} \subseteq Q^{m-1}_{m-1,1/2^{m-1}} \times [0, 1]^{m+1}$.
- A vector $\vec{z} \in \mathbb{A}$ belongs to $Q^m_{m,1/2^m} \times \prod_{j=m+1}^{\infty}[0, 1]^{j+1}$ if and only if $p_i(z) \in [l_i, u_i]$ for every element (p_i, l_i, u_i) of the sequence P^m.

Fix an integer $m \in \mathbb{N}$ and suppose that we have defined bounding sequences $P^0, \ldots, P^{m-1}$ and stabilizing systems $S^0, \ldots, S^{m-1}$. By Lemma 14, there exists a stabilizing system S' for t_k^{m-1} and the same b_k, $k \in \mathbb{N}$, such that $\|w'_k(x) - w_k^{m-1}(x)\|_\infty < 1/2^{m+1}$ for any $x \in V_{k-1}^{m-1} \times U_k^{m-1}$ with $k \in \mathbb{N}$; if $m = 1$, we just set S' to be S^0. We next apply Lemma 17 with $c = 1/2 + 1/2^m$, $c' = 1/2 + 1/2^{m+1}$ and $\varepsilon = 1/2^{m+1}$ to obtain an m-excellent stabilizing system S^m for t_k^{m-1}, $k \in \mathbb{N}$, which is m-finite and $(1/2+1/2^{m+1})$-strong. Also note that $\|b_k^m - b_k^{m-1}\|_\infty < 1/2^{m+1}$ for every $k \in \mathbb{N}$, and $(J_k^{m-1})_{k\in\mathbb{N}} \preceq (J_k^m)_{k\in\mathbb{N}}$. Furthermore, if k_0 is the largest integer such that $J_k^m = J_k^{m-1}$ for all $k \in [k_0]$, then $I_k^m = I_k^{m-1}$, $d_k^m = d_k^{m-1}$, $U_k^m = U_k^{m-1}$, $V_k^m \subseteq V_k^{m-1}$, and

$$\begin{aligned}\|w_k^m(x) - w_k^{m-1}(x)\|_\infty &\le \|w_k^m(x) - w'_k(x)\|_\infty + \|w'_k(x) - w_k^{m-1}(x)\|_\infty \\ &< 1/2^{m+1} + 1/2^{m+1} = 1/2^m\end{aligned}$$

for every $x \in V_k^m$ and $k \in [k_0]$. This implies $Q_{k_0,1/2^m}^m \subseteq Q_{k_0,1/2^{m-1}}^{m-1}$. For $k > k_0$, we have that $||\widetilde{w}_k^m(x) - b_k^m||_\infty < 1/2^{m+1}$ for all $x \in V_{k-1}^m \times U_k^m$ and $||b_k^m - b_k^{m-1}||_\infty < 1/2^{m+1}$. It follows that $Q_{m,1/2^m}^m \subseteq Q_{m-1,1/2^{m-1}}^{m-1} \times [0,1]^{m+1}$. This finishes the definition of the stabilizing system S^m and verifies its properties as stated above.

We next define the bounding sequence P^m. Set K to be the maximum integer obtained when Lemma 11 is applied with P^{m-1}, $\varepsilon = \delta_m$ and each of the graphs $G_1, \ldots, G_m$. Choose an infinite increasing sequence k_i, $i \in \mathbb{N}$, with $k_1 \ge K$, such that $P_{k_i}^{m-1} = (0,0,1)$ for every $i \in \mathbb{N}$ and there exists an infinite number of indices $j \in \mathbb{N}$ such that $P_j^{m-1} = (0,0,1)$ and $j \ne k_i$ for all $i \in \mathbb{N}$. The Stone-Weierstrass Theorem implies that, for every $i \in \mathbb{N}$, there exists a polynomial $q_i(z_1, \ldots, z_m)$, $z_j \in [0,1]^{j+1}$, $j \in [m]$, such that

- $q_i(z_1, \ldots, z_m) \in [0,1]$ for every $(z_1, \ldots, z_m) \in [0,1]^{2+\cdots+(m+1)}$,
- $q_i(z_1, \ldots, z_m) \in [1-2^{-i}, 1]$ for every $(z_1, \ldots, z_m) \in Q_{m,1/2^m}^m$, and
- $q_i(z_1, \ldots, z_m) \in [0, 2^{-i}]$ for $(z_1, \ldots, z_m)$ at Hausdorff distance at least 2^{-i} from $Q_{m,1/2^m}^m$.

Let $\xi_i \in (0,1]$ be a real such that all coefficients of $\xi_i \cdot q_i$ have absolute value at most $2^{-2^{k_i}}/9$ and all the partial derivatives of $\xi_i \cdot q_i$ belong to $[-1,+1]$ on $[0,1]^{2+\cdots+(m+1)}$. The bounding sequence P^m is defined as follows: $P_{k_i}^m$ is $(\xi_i \cdot q_i, \xi_i(1-2^{-i}), \xi_i)$ for $i \in \mathbb{N}$, and P_j^m is P_j^{m-1} for the remaining indices j, i.e., if $j \ne k_i$ for all $i \in \mathbb{N}$. Observe that $(z_1, \ldots, z_m) \in [0,1]^{2+\cdots+(m+1)}$ belongs to $Q_{m,1/2^m}^m$ if and only if $p_i(z_1, \ldots, z_m) \in [l_i, u_i]$ for every element (p_i, l_i, u_i) of the sequence P^m. Indeed, the construction implies that $Q_{m,1/2^m}^m \subseteq Q_{m-1,1/2^{m-1}}^{m-1} \times [0,1]^{m+1}$; in particular, any point in $Q_{m,1/2^m}^m$ satisfies all constraints implied by P^{m-1}. Moreover, the functions t_k^m and t_k^{m-1} for $k \le m$ are δ_m-close.

We have now defined the bounding sequence P^m and the stabilizing system S^m for every $m \in \mathbb{N}$. Since each element of the bounding sequences P^m changes

at most once during the iterative procedure described above, we can define a bounding sequence P as
$$P_i = \lim_{m\to\infty} P_i^m .$$
The bounding sequence P determines the family $\mathcal{W}_P$ of graphons; the family $\mathcal{W}$ from the statement of the theorem will be constructed as a subset of $\mathcal{W}_P$.

Recall that $(J_k^{m-1})_{k\in\mathbb{N}} \preceq (J_k^m)_{k\in\mathbb{N}}$ for every $m \in \mathbb{N}$. This implies that for every $k \in \mathbb{N}$, there exists m_0 such that J_k^m is the same for all $m \geq m_0$, and consequently I_k^m, d_k^m and U_k^m are the same and $V_k^{m+1} \subseteq V_k^m$ for all $m \geq m_0$. Hence, we can define
$$J_k = \lim_{m\to\infty} J_k^m,\ \ d_k = \lim_{m\to\infty} d_k^m \text{ and } U_k = \lim_{m\to\infty} U_k^m .$$
Likewise, we can define
$$V_k = \bigcap_{m\in\mathbb{N}\setminus[k]} V_k^m \text{ and } V = \bigcap_{k\in\mathbb{N}} \left(V_k \times [0,1]^{\mathbb{N}\setminus[k]}\right) .$$
Since each V_k^m satisfies for every $(x_1,\ldots,x_{k-1}) \in V_{k-1}^m$ that the measure of $x_k \in U_k^m$ such that $(x_1,\ldots,x_k) \in V_k^m$ is at least $(1/2 + 1/2^{m+1})\,|U_k^m|$, it follows that V_k satisfies for every $(x_1,\ldots,x_{k-1}) \in V_{k-1}$ that the measure of $x_k \in U_k$ such that $(x_1,\ldots,x_k) \in V_k$ is at least $|U_k|/2$.

Since the sequence $(b_k^m)_{m\in\mathbb{N}}$ is Cauchy for every $k \in \mathbb{N}$, we can also define b_k to be the limit of the sequence $(b_k^m)_{m\in\mathbb{N}}$. Finally, the sequence of functions $(w_k^m)_{m\in\mathbb{N}}$ is uniformly convergent on $V_{k-1} \times U_k$, and we set w_k to be the limit. In this way, we have defined a stabilizing system $S = (U_k, V_k, I_k, J_k, d_k, b_k, w_k)_{k\in\mathbb{N}}$. Observe that S is 1/2-strong. Also observe that
$$w_k(x, b_{k,d_k})_j = \lim_{m\to\infty} w_k(x, b_{k,d_k}^m)_j = \lim_{m\to\infty} w_k^m(x, b_{k,d_k}^m)_j = \lim_{m\to\infty} b_{k,j}^m = b_{k,j}.$$
for every $x \in V_{k-1}$ and $j \in J_k$, where for the second equality we used the fact that w_k^m converges uniformly to w_k. It follows that $\widetilde{w}_k(x, b_{k,d_k}) = b_k$ for every $x \in V_{k-1}$. Finally, observe that $z \in [0,1]^{\mathbb{N}}$ satisfies $p_i(z) \in [l_i, u_i]$ for every element (p_i, l_i, u_i) of the sequence P if and only if there exist $x \in V$ such that $\vec{z} = (\widetilde{w}_1(x_1), \widetilde{w}_2(x_1,x_2), \ldots)$.

Let t_k be the function t_{P,G_k} and observe that the sequence $(t_k^m)_{m\in\mathbb{N}}$ uniformly converges to t_k on the set containing all $z \in [0,1]^{\mathbb{N}}$ such that $p(z) \in [l,u]$ for every element (p,l,u) of the sequence P. Consequently, we obtain that
$$t_i(\widetilde{w}_{\leq k}(x), b_{>k}) = \lim_{m\to\infty} t_i(\widetilde{w}_{\leq k}^m(x), b_{>k}) = \lim_{m\to\infty} t_i^m(\widetilde{w}_{\leq k}^m(x), b_{>k})$$
for every $i \in [k]$, $x \in V_k$ and $k \in \mathbb{N}$, where we used that t_i^m converges uniformly to t_i. Using Lemma 15, this implies that
$$t_i(\widetilde{w}_{\leq k}(x), b_{>k}) = \lim_{m\to\infty} t_i^m(\widetilde{w}_{\leq k}^m(x), b_{>k}) = \lim_{m\to\infty} t_i^m(\widetilde{w}_{\leq k-1}^m(x_1,\ldots,x_{k-1}), b_k, b_{>k})$$

$$=t_i(\widetilde{w}_{\le k-1}(x_1,\dots,x_{k-1}),b_k,b_{>k})$$

for every $i\in[k]$, $x\in V_k$ and $k\in\mathbb{N}$. It follows that

$$t_i(\widetilde{w}_1(x_1),\widetilde{w}_2(x_1,x_2),\dots)=t_i(\widetilde{w}_{\le i-1}(x_1,\dots,x_{i-1}),b_{\ge i})$$

for every $x\in V$. Thus, if two elements x and x' from V agree on the first $i-1$ coordinates, then $t_i(\widetilde{w}_1(x_1),\widetilde{w}_2(x_1,x_2),\dots)$ and $t_i(\widetilde{w}_1(x'_1),\widetilde{w}_2(x'_1,x'_2),\dots)$ are the same, i.e.,

$$\tau(G_i,W_P(\widetilde{w}_1(x_1),\widetilde{w}_2(x_1,x_2),\dots))=\tau(G_i,W_P(\widetilde{w}_1(x'_1),\widetilde{w}_2(x'_1,x'_2),\dots)).$$

Consequently, if x and x' from V agree on the first $r-1$ coordinates, then the densities of all graphs $G_1,\dots,G_r$ in the graphons $W_P(\widetilde{w}_1(x_1),\widetilde{w}_2(x_1,x_2),\dots)$ and $W_P(\widetilde{w}_1(x'_1),\widetilde{w}_2(x'_1,x'_2),\dots)$ are the same.

We are now ready to define the family $\mathcal{W}$ of graphons with the properties from the statement of the theorem. Theorem 9 implies that there exists graphs $H_1,\dots,H_\ell$ and a polynomial q in ℓ variables such that a graphon W is weakly isomorphic to a graphon contained in $\mathcal{W}_P$ if and only if $q(d(H_1,W),\dots,d(H_\ell,W))=0$. Set $d_i=d(H_i,W_P(b_1,b_2,\dots))$ for $i\in[\ell]$ and define the family $\mathcal{W}$ as follows

$$\mathcal{W}=\{W\text{ such that }d(H_i,W)=d_i\text{ for all }i\in[\ell]\}.$$

Observe that $\mathcal{W}\subseteq\mathcal{W}_P$.

Suppose that a graphon $W\in\mathcal{W}$ is finitely forcible. We can assume that there exist $r\in\mathbb{N}$ and $d'_1,\dots,d'_r$ such that W is the unique graphon in $\mathcal{W}$ with the density of G_i equal to d'_i for $i\in[r]$, and the graphs $G_1,\dots,G_r$ include all graphs $H_1,\dots,H_\ell$. Since W is weakly isomorphic to a graphon in $\mathcal{W}_P$, there exists $x\in V$ such that W and $W_P(\widetilde{w}_1(x_1),\widetilde{w}_2(x_1,x_2),\dots)$ are weakly isomorphic. Since the measure of y_r such that $(x_1,\dots,x_{r-1},y_r)\in V_r$ is at least $|U_r|/2>0$, there exists $x'_r\in V_r$ such that $x'_r\ne x_r$ and $(x_1,\dots,x_{r-1},x'_r)\in V_r$. We set $x'_i=x_i$ for $i\in[r-1]$ and iteratively find x'_i for $i>r$ as follows: if $x'_1,\dots,x'_{i-1}$ has been constructed for some $i>r$, then we choose x'_i to be an arbitrary element of U_i such that $(x'_1,\dots,x'_{i-1},x'_i)\in V_i$; note that such x'_i exists since the measure of suitable choices of x'_i is at least $|U_i|/2>0$. Let W' be the graphon $W_P(\widetilde{w}_1(x'_1),\widetilde{w}_2(x'_1,x'_2),\ldots)$. Since $x'\in V$, the graphon W' belongs to the family $\mathcal{W}_P$. In addition, the density of each of the graphs $G_1,\dots,G_r$ in W and in W' is the same. On the other hand, W' is not weakly isomorphic to W by Theorem 9 because $\widetilde{w}_r(x_1,\dots,x_r)\ne\widetilde{w}_r(x'_1,\dots,x'_r)$. We conclude that the graphon W is not finitely forcible. $\square$

6 Finitely forcible graphon families

In this section, we define the family of graphons $W_P(z)$, and prove Theorem 9. The general structure of graphons $W_P(z)$ is visualized in Figure 3. The graphon

Part	A	B	C	D_A	$\cdots$	D_G	E	F	Q	R
Degree	$\frac{1201}{2500}$	$\frac{1202}{2500}$	$\frac{1203}{2500}$	$\frac{1204}{2500}$	$\cdots$	$\frac{1210}{2500}$	$\frac{1211}{2500}$	$\frac{1212}{2500}$	$> \frac{1300}{2500}$	$\frac{1278}{2500}$

Table 1: The degrees of the parts of the graphon $W_P(z)$.

$W_P(z)$ is a partitioned graphon with 14 parts, denoted A, B, C, D_A, D_B, $\ldots$, D_G, E, F, Q and R. The size of each part besides Q is $1/25$, and the size of Q is $12/25$. The degree of each part is given in Table 1 (we do not compute the exact value of the degree of the part Q). Each part A, B, C, D_A, D_B, $\ldots$, D_G, E, F, Q and R is a half-open subinterval of $[0, 1)$.

Let J_k, $k \in \mathbb{N}$, be the interval $[1 - 2^{-k+1}, 1 - 2^{-k})$, so $J_1 = [0, 1/2)$, $J_2 = [1/2, 3/4)$, $J_3 = [3/4, 7/8)$, etc. Note that the intervals J_k, $k \in \mathbb{N}$, give a partition of $[0, 1)$. If $x \in \mathbb{R}$, define $[\![x]\!]$ as the unique integer k such that $x - \lfloor x \rfloor \in J_k$.

We start by defining an auxiliary graphon W_F, which is visualized in Figure 4. Fix a bounding sequence $P = (p_i, l_i, u_i)_{i\in\mathbb{N}}$, and recall that $\pi_{i,j}$ is the coefficient of the monomial z^{M_j} in the polynomial p_i. Let $\pi^+_{i,j}$ be equal to $\pi_{i,j}$ if $\pi_{i,j} \geq 0$, and 0 otherwise; similarly, let $\pi^-_{i,j}$ be equal to $-\pi_{i,j}$ if $\pi_{i,j} < 0$, and 0 otherwise. Note that the definition of a bounding sequence implies that $|\pi_{i,j}| \leq 2^{-2^j}/9 \leq 2^{-2j}/9$ for all $i, j \in \mathbb{N}$.

The value of $W_F(x, y)$ is equal to 1 for $(x, y) \in [0, 2/3)^2 \setminus [1/3, 2/3)^2$ if

- $x, y \in [1/6, 1/3)$ and $[\![3x]\!] = [\![3y]\!]$,
- $x \in [1/3, 2/3)$, $y \in [1/6, 1/3)$, and either $M_{[\![3x]\!]} = M_{[\![3y]\!]} \setminus \{\min M_{[\![3y]\!]}\}$ or $M_{[\![3x]\!]} = \{\min M_{[\![3y]\!]}\}$, or
- $y \in [1/3, 2/3)$, $x \in [1/6, 1/3)$, and either $M_{[\![3y]\!]} = M_{[\![3x]\!]} \setminus \{\min M_{[\![3x]\!]}\}$ or $M_{[\![3y]\!]} = \{\min M_{[\![3x]\!]}\}$.

Recall that both $M_{[\![3x]\!]}$ and $M_{[\![3y]\!]}$ are multisets and $M_{[\![3y]\!]} \setminus \{\min M_{[\![3y]\!]}\}$ is the multiset obtained from $M_{[\![3y]\!]}$ by removing a single instance of $\min M_{[\![3y]\!]}$. The value of $W_F(x, y)$ is equal to 0 everywhere else on $(x, y) \in [0, 2/3)^2 \setminus [1/3, 2/3)^2$. The value of $W_F(x, y)$ for the remaining $(x, y) \in [0, 1)^2$ is defined as follows:

$$W_F(x,y) = \begin{cases} 9 \cdot 2^{2[\![3y]\!]} \cdot \pi^+_{[\![3x]\!],[\![3y]\!]} & \text{if } x \in [2/3, 1) \text{ and } y \in [0, 1/3), \\ 9 \cdot 2^{2[\![3y]\!]} \cdot \pi^-_{[\![3x]\!],[\![3y]\!]} & \text{if } x \in [2/3, 1) \text{ and } y \in [1/3, 2/3), \\ 9 \cdot 2^{2[\![3x]\!]} \cdot \pi^+_{[\![3y]\!],[\![3x]\!]} & \text{if } x \in [0, 1/3) \text{ and } y \in [2/3, 1), \\ 9 \cdot 2^{2[\![3x]\!]} \cdot \pi^-_{[\![3y]\!],[\![3x]\!]} & \text{if } x \in [1/3, 2/3) \text{ and } y \in [2/3, 1), \\ 1 - u_{[\![3x]\!]} & \text{if } x, y \in [1/3, 2/3) \text{ and } [\![3x]\!] = [\![3y]\!], \\ 0 & \text{if } x, y \in [1/3, 2/3) \text{ and } [\![3x]\!] \neq [\![3y]\!], \\ l_{[\![3x]\!]} & \text{if } x, y \in [2/3, 1) \text{ and } [\![3x]\!] = [\![3y]\!], \text{ and} \\ 0 & \text{if } x, y \in [2/3, 1) \text{ and } [\![3x]\!] \neq [\![3y]\!]. \end{cases}$$

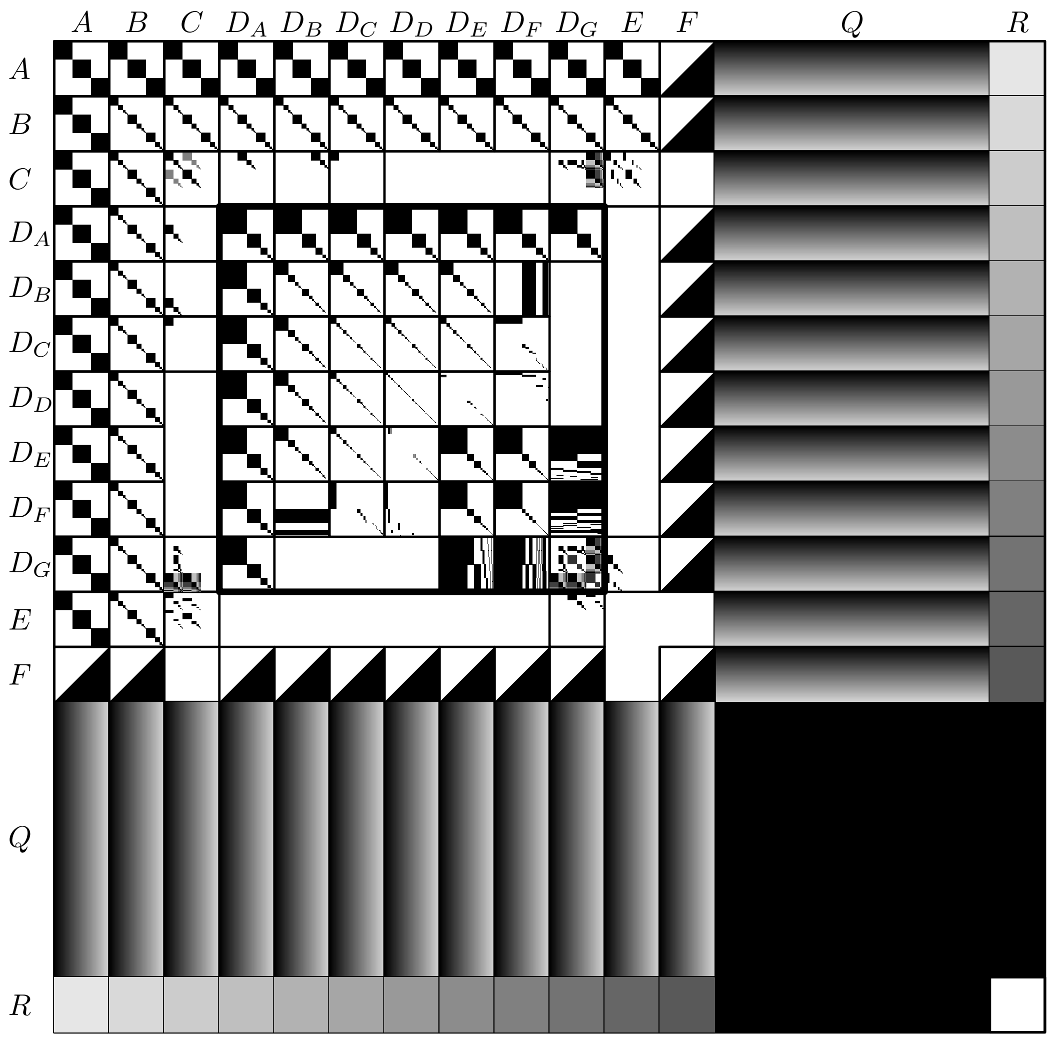

Figure 3: The structure of the graphon $W_P(z)$. The parts of the graphon corresponding to the graphon W_0 from Theorem 2 are $D_A, \ldots, D_G$ and are framed by a thicker line. Note that the size of the part Q is not drawn to scale.

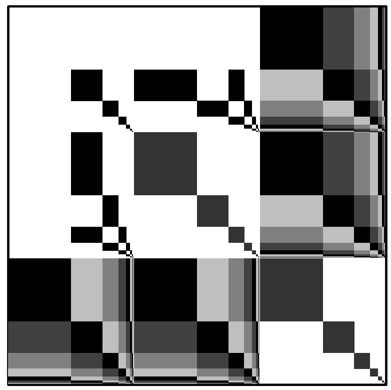

Figure 4: Visualization of the graphon W_F, which is in the tile $D_G \times D_G$ of the graphon $W_P(z)$.

By Theorem 2, there exists a finitely forcible graphon W_0 such that W_F is a subgraphon of W_0. The graphon W_0, which is depicted in Figure 1, is a partitioned graphon with 10 parts $A, \ldots, G, P, Q, R$. Let ι^0_X be the linear map from $[0,1)$ to the part X of the graphon W_0. Since the graphon W_F is embedded in the tile $G \times G$ of W_0, it holds that $W_0(\iota^0_G(x), \iota^0_G(y)) = W_F(x,y)$ for all $(x,y) \in [0,1)^2$. Also note that the definition of the graphon W_0 depends on P only, i.e., it does not depend on $z \in [0,1]^{\mathbb{N}}$.

We are now ready to define the graphon $W_P(z)$. Fix $z \in [0,1]^{\mathbb{N}}$ (in addition to the sequence P, which we have already fixed). For the rest of this section, we will just write W_P for $W_P(z)$, which allows us to write $W_P(x,y)$ for the value of the graphon $W_P(z)$ for $(x,y) \in [0,1)^2$. For each part X of W_P, let ι_X be the linear map from $[0,1)$ to X. For $Z, Z' \in \{A, \ldots, G\}$, set $W_P(\iota_{D_Z}(x), \iota_{D_{Z'}}(y)) = W_0(\iota^0_Z(x), \iota^0_{Z'}(y))$ for all $(x,y) \in [0,1)^2$. The values of W_P on the remaining pairs of the parts A, B, C, D_A, D_B, $\ldots$, D_G, E, and F are given in Table 2.

The description of the tiles $C \times E$ and $D_G \times E$ may seem harder to immediately grasp. Let us examine e.g. the condition $0 \le 2^{[\![3x]\!]}(3y-1) + 2 \le p_{[\![3x]\!]}(z)$ in the first case in the description of the tile $C \times E$. Observe that the condition is equivalent to

$$y \in \left[\frac{1 - 2^{-([\![3x]\!]-1)}}{3}, \frac{1 - 2^{-([\![3x]\!]-1)} + 2^{-[\![3x]\!]} \cdot p_{[\![3x]\!]}(z)}{3}\right].$$

In particular, the condition can only be satisfied if $[\![3x]\!] = [\![3y]\!]$. The other conditions in the description of the tile $C \times E$ and two of the conditions in the description of the tile $D_G \times E$ can be expressed analogously.

Tile $X \times Y$	$W_P(\iota_X(x), \iota_Y(y))$	Domain of (x,y)	Condition
$A \times A$ $\vdots$ $A \times E$	1	$[0,1)^2$	$\lfloor 3x \rfloor = \lfloor 3y \rfloor$
$B \times B$ $\vdots$ $B \times E$	1	$[0,1/3)^2 \cup$ $[1/3,2/3)^2 \cup$ $[2/3,1)^2$	$\llbracket 3x \rrbracket = \llbracket 3y \rrbracket$
$C \times C$	$z^{M_{\llbracket 3x \rrbracket}}$	$[0,1/3)^2$	$\llbracket 3x \rrbracket = \llbracket 3y \rrbracket$
	$z^{M_{\llbracket 3x \rrbracket}}$	$[1/3,2/3)^2$	$\llbracket 3x \rrbracket = \llbracket 3y \rrbracket$
	$1 - z^{M_{\llbracket 3x \rrbracket}}$	$[0,1/3) \times [1/3,2/3)$	$\llbracket 3x \rrbracket = \llbracket 3y \rrbracket$
	$1 - z^{M_{\llbracket 3x \rrbracket}}$	$[1/3,2/3) \times [0,1/3)$	$\llbracket 3x \rrbracket = \llbracket 3y \rrbracket$
$C \times D_A$	1	$[0,1/3) \times [1/3,2/3)$	$\llbracket 3x \rrbracket = \llbracket 3y \rrbracket$
$C \times D_B$	1	$[0,1/3) \times [2/3,1)$	$\llbracket 3x \rrbracket = \llbracket 3y \rrbracket$
$C \times D_C$	1	$[0,1/6)^2$	
$C \times D_G$	$W_F(x,y)$	$[0,1/3) \times [0,2/3) \cup$ $[0,2/3) \times [2/3,1)$	
$C \times E$	1	$[0,1/3) \times [0,1/3)$	$0 \leq 2^{\llbracket 3x \rrbracket}(3y-1)+2 \leq p_{\llbracket 3x \rrbracket}(z)$
	1	$[0,1/3) \times [1/3,2/3)$	$0 \leq 2^{\llbracket 3x \rrbracket}(3y-2)+2 \leq 1 - p_{\llbracket 3x \rrbracket}(z)$
	1	$[1/3,2/3) \times [0,1/3)$	$1 \geq 2^{\llbracket 3x \rrbracket}(3y-1)+2 \geq p_{\llbracket 3x \rrbracket}(z)$
	1	$[1/3,2/3) \times [1/3,2/3)$	$1 \geq 2^{\llbracket 3x \rrbracket}(3y-2)+2 \geq 1 - p_{\llbracket 3x \rrbracket}(z)$
$D_G \times E$	1	$[1/3,2/3) \times [0,1/3)$	$\llbracket 3x \rrbracket = \llbracket 3y \rrbracket$ and $2^{\llbracket 3y \rrbracket}(3x-1)+2 \leq u_{\llbracket 3y \rrbracket}$
	1	$[2/3,1) \times [0,1/3)$	$\llbracket 3x \rrbracket = \llbracket 3y \rrbracket$ and $2^{\llbracket 3y \rrbracket}(3x-1)+2 \leq l_{\llbracket 3y \rrbracket}$
$A \times F$ $B \times F$ $D_A \times F$ $\vdots$ $D_G \times F$ $F \times F$	1	$[0,1)^2$	$x + y \geq 1$

Table 2: The definition of the values $W_P(x,y)$ for x, y from the parts A, B, C, D_A, D_B, $\ldots$, D_G, E, and F except for the tiles $D_Z \times D_{Z'}$, $Z, Z' \in \{A, \ldots, G\}$. The definition for symmetric pairs of tiles is omitted, e.g., the values on the tile $A \times B$ define the values on the tile $B \times A$. In each case, the tile is set to 0 if none of the conditions are satisfied.

We now set $W_P(\iota_X(x), \iota_Q(y))$ to be equal to the following integral

$$W_P(\iota_X(x), \iota_Q(y)) = \frac{1}{12}\left(12 - \sum_{Z \in \{A,B,C,D_A,\ldots,D_G,E,F\}} \int_{[0,1)} W_P(\iota_X(x), \iota_Z(z))\,\mathrm{d}z\right)$$

for all $X \in \{A, B, C, D_A, \ldots, D_G, E, F\}$ and all $x, y \in [0, 1)$. Note that the sum above has 12 terms, so its value is always between 0 and 1 (inclusively). Finally, we set W_P to be equal to 1 on the tiles $Q \times Q$ and $Q \times R$, to $1/25, \ldots, 12/25$ on the tiles $A \times R$, $B \times R$, $C \times R$, $D_A \times R$, $D_B \times R$, $\ldots$, $D_G \times R$, $E \times R$, and $F \times R$, respectively, and to zero on the tile $R \times R$. This finishes the definition of the graphon W_P.

We next prove the following lemma on the structure of the just defined graphons $W_P(z)$.

Lemma 19. *Let $P = (p_i, l_i, u_i)_{i\in\mathbb{N}}$ be a bounding sequence. For any $z, z' \in [0, 1]^{\mathbb{N}}$, the graphons $W_P(z)$ and $W_P(z')$ are the same everywhere except for the tiles $C\times C$ and $C \times E$.*

Proof. Inspecting the definition of the graphon $W_P(z)$, it is easy to notice that the tiles $C\times C$ and $C\times E$ are indeed the only tiles among the tiles $X\times Y$, $X, Y \in \{A, B, C, D_A, \ldots, D_G, E, F, R\}$ with the structure depending on the choice of z. However, the changes inside the tiles $C \times C$ and $C \times E$ could possibly result in a change of the tile $C \times Q$ or $E \times Q$. This can happen only if there exists $x \in C$ such that the values of $\deg^C_{W_P(z)}(x)$ and $\deg^E_{W_P(z)}(x)$ are not constant as z varies, or there exists $x \in E$ such that the value of $\deg^C_{W_P(z)}(x)$ varies. Observe, however, that the following holds:

$$\begin{aligned}
\deg^C_{W_P(z)}(\iota_C(x)) &= \begin{cases} 2^{-[\![3x]\!]}/3 & \text{for } x \in [0, 2/3), \\ 0 & \text{for } x \in [2/3, 1), \end{cases} \\
\deg^E_{W_P(z)}(\iota_C(x)) &= \begin{cases} 2^{-[\![3x]\!]}/3 & \text{for } x \in [0, 2/3), \\ 0 & \text{for } x \in [2/3, 1), \end{cases} \\
\deg^C_{W_P(z)}(\iota_E(x)) &= \begin{cases} 2^{-[\![3x]\!]}/3 & \text{for } x \in [0, 2/3), \\ 0 & \text{for } x \in [2/3, 1). \end{cases}
\end{aligned}$$

Hence, none of the relative degrees depend on z, which implies that the tiles $C \times Q$ and $E \times Q$ also do not depend on z. ☐

Lemma 19 implies that for each $X \in \{A, B, C, D_A, \ldots, D_G, E, F, Q, R\}$, the tile $X \times Q$ is the same in all graphons $W_P(z)$, $z \in [0, 1]^{\mathbb{N}}$, i.e., the tile is independent of the choice of z. In particular, the degree of the part Q does not depend on $z \in [0, 1]^{\mathbb{N}}$.

We next prove the following claim about the graphons $W_P(z)$.

Lemma 20. *Let P be a bounding sequence. The graphons $W_P(z)$ and $W_P(z')$, $z, z' \in [0, 1]^{\mathbb{N}}$, are weakly isomorphic if and only if $z = z'$.*

Proof. Fix $z \in [0,1]^{\mathbb{N}}$. In $W_P(z)$, the vertices of degree 1203/2500 uniquely determine the part C of the graphon. The part C can be uniquely split into disjoint measurable subsets C_i, $i \in \mathbb{N}_0$ with $|C_0| = 1/3$ and $|C_i| = 2^{-i+1}/3$ as follows. If $x \in C_0$, then $\deg_W^C(x) = 0$, and if $x \in C_i$ for $i \in \mathbb{N}$, then $\deg_W^C(x) = 2^{-i}/3$. Inside the tile $C \times C$, the graphon is non-zero only on the sets $C_i \times C_i$, $i \in \mathbb{N}$. The structure of W_P restricted to $C_i \times C_i$ uniquely determines the value of z^{M_i}. In particular, the values of all z_i, $i \in \mathbb{N}$, are uniquely determined in this way. It follows that if two graphons $W_P(z)$ and $W_P(z')$ are weakly isomorphic, then $z = z'$. □

The remainder of this section is devoted to the proof of the following theorem.

Theorem 21. *There exist graphs $H_1, \ldots, H_\ell$ and an integer D with the following property. For any bounding sequence $P = (p_i, l_i, u_i)_{i\in\mathbb{N}}$, there exists a polynomial q of degree at most D in ℓ variables such that the following two statements are equivalent for any graphon W.*

- *The graphon W is weakly isomorphic to a graphon $W_P(z)$ for some $z \in [0,1]^{\mathbb{N}}$ such that $p_i(z) \in [l_i, u_i]$ for all $i \in \mathbb{N}$.*
- *It holds that $q(d(H_1, W), \ldots, d(H_\ell, W)) = 0$.*

6.1 Proof of Theorem 21 – general setting

Theorem 21 follows from the following statement: there exists a family $\mathcal{C}$ of ordinary and decorated density constraints such that

- the graphs appearing in $\mathcal{C}$ do not depend on P, and
- a graphon W satisfies all constraints in $\mathcal{C}$ if and only if W is weakly isomorphic to a graphon $W_P(z)$ for some $z \in [0,1]^{\mathbb{N}}$, with $p_i(z) \in [l_i, u_i]$ for all $i \in \mathbb{N}$.

Indeed, if we find such a family $\mathcal{C}$, Lemma 5 would imply that there exists a family $\mathcal{C}'$ of ordinary density constraints with the same properties. Each constraint in $\mathcal{C}'$ can be thought of as being a polynomial p in the densities of graphs appearing in the constraint such that $p = 0$ if and only if the constraint is satisfied. The sought polynomial q can then be set to be the sum of p^2 taken over all constraints in $\mathcal{C}$. Hence, we focus on finding a family $\mathcal{C}$ with the above properties.

The constraints forming the family $\mathcal{C}$ will be presented in this and the following subsections together with the related parts of the proof of Theorem 21. The family $\mathcal{C}$ contains the ordinary density constraints that are satisfied precisely by graphons with the same part sizes and degrees as graphons $W_P(z)$, $z \in [0,1]^{\mathbb{N}}$; such ordinary density constraints exist by Lemma 5. Note that the sizes and the degrees of the parts do not depend on $z \in [0,1]^{\mathbb{N}}$ (see the remark after Lemma 19).

Suppose that W is a graphon satisfying all constraints in $\mathcal{C}$. Then W is a partitioned graphon with parts corresponding to those of W_P, and we will write $A, B, C, D_A, \dots, D_G, E, F, Q, R$ for the parts of W. We will show that there exists a choice of $z \in [0,1]^{\mathbb{N}}$ satisfying all constraints implied by the bounding sequence P such that W and $W_P(z)$ are weakly isomorphic. To do so, we will show that there exist a vector $z \in [0,1]^{\mathbb{N}}$ and a measure preserving map $g : [0,1) \to [0,1)$ such that $W(x,y) = W_P(z)(g(x), g(y))$ for almost all $(x,y) \in [0,1)^2$.

Let A^P, B^P, C^P, D^P_A, $\dots$, D^P_G, E^P, F^P, Q^P, R^P be the half-open subintervals of $[0,1)$ forming the parts of the graphon W_P (we use the superscripts to make a clear distinction between the parts of W and the parts of W_P). The Monotone Reordering Theorem [32, Proposition A.19] implies that for every $X \in \{A, B, D_A, \dots, D_G, F, Q, R\}$, there exist a measure preserving map $\varphi_X : X \to [0, |X|)$ and a non-decreasing function $f_X : [0, |X|) \to \mathbb{R}$ such that

$$f_X(\varphi_X(x)) = \deg^F_W(x) = \frac{1}{|F|} \int_F W(x,y)\,\mathrm{d}y$$

for almost every $x \in X$. In addition, there exist measure preserving maps $\varphi_C : C \to [0, 1/25)$ and $\varphi_E : E \to [0, 1/25)$ and non-decreasing functions $f_C : [0, 1/25) \to \mathbb{R}$ and $f_E : [0, 1/25) \to \mathbb{R}$ such that

$$f_C(\varphi_C(x)) = 900 \int_{N^A_W(x)} \deg^F_W(z)\,\mathrm{d}z - \deg^B_W(x) \qquad \text{and} \tag{4}$$

$$\begin{aligned} f_E(\varphi_E(x')) \quad &= \quad 4500 \int_{N^A_W(x')} \deg^F_W(z)\,\mathrm{d}z - 5\deg^B_W(x') - \deg^{D_G}_W(x') \\ &\qquad + \int_{N^C_W(x')} \deg^{D_A}_W(z)\,\mathrm{d}z \end{aligned} \tag{5}$$

for almost every $x \in C$ and $x' \in E$.

For $x \in X$, where $X \in \{A, B, C, D_A, \dots, D_G, E, F, Q, R\}$, we set $g(x)$ to $\inf X^P + \varphi_X(x)$. Since each X^P is a subinterval of length of $|X|$, g is a measure preserving map from $[0,1)$ to $[0,1)$. In Subsections 6.2–6.7, we will show that $W(x,y) = W_P(g(x), g(y))$ almost everywhere for a suitable choice of $z \in [0,1]^{\mathbb{N}}$. Finally, for $X \in \{A, B, C, D_A, \dots, D_G, E, F, Q, R\}$, define $\eta_X : X \to [0,1)$ as $\eta_X(x) = \varphi(X)/|X|$. Note that $\iota_X(\eta_X(x)) = g(x)$ for $x \in X$. The mutual relations of the just defined maps are visualized in Figure 5.

6.2 Proof of Theorem 21 – coordinate system

The half-graphon W_Δ is the graphon such that $W_\Delta(x,y) = 1$ if $x + y \geq 1$ and $W_\Delta(x,y) = 0$, otherwise. The half-graphon W_Δ is finitely forcible [17, 34]. By Lemma 6, there exists a collection of decorated constraints that is satisfied if and

$$\begin{tikzcd} & {[0,|X|)} \arrow[dr, "+\inf X^P"] & \\ X \arrow[ur, "\varphi_X"] \arrow[rr, "g"] \arrow[dr, "\eta_X"'] & & X^P \\ & {[0,1)} \arrow[ur, "\iota_X"'] & \end{tikzcd}$$

Figure 5: The structure of maps used in the proof of Theorem 21.

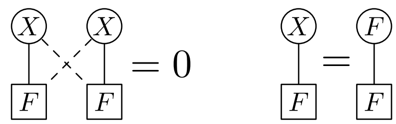

Figure 6: Decorated constraints forcing the structure of the tiles $X \times F$, where $X \in \{A, B, D_A, \dots, D_G\}$.

only if the tile $F \times F$ is weakly isomorphic to the half-graphon W_Δ. The definitions of φ_F and f_F then imply that for almost all $(x, y) \in F \times F$, $W(x, y) \in \{0, 1\}$, and $W(x, y) = 1$ iff $f_F(\varphi_F(x)) + f_F(\varphi_F(y)) \geq 1$. It follows that $W(x, y) = W_P(g(x), g(y))$ for almost all $(x, y) \in F \times F$ and $f_F(z) = 25z$ for almost every $z \in [0, 1/25)$.

Fix $X \in \{A, B, D_A, \dots, D_G\}$ and consider the two decorated constraints depicted in Figure 6. The first constraint implies that $W(x, y) \in \{0, 1\}$ for almost all $(x, y) \in X \times F$. In particular, for almost all $y \in F$, it holds that $W(x, y) \in \{0, 1\}$ for almost all $x \in X$. Furthermore, for almost every pair $y, y' \in F$, we have $N_X(y) \sqsubseteq N_X(y')$ or $N_X(y') \sqsubseteq N_X(y)$, which implies that for almost all $y \in F$, it holds that for almost every $y' \in F$, we have $N_X(y) \sqsubseteq N_X(y')$ or $N_X(y') \sqsubseteq N_X(y)$. The second constraint implies that $\deg^X(y) = \deg^F(y) = f_F(\varphi_X(y)) = 25\varphi_X(y)$ for almost every $y \in F$. Let F' consist of those points in $y \in F$ for which

- $W(x, y) \in \{0, 1\}$ for almost all $x \in X$,
- $N_X(y) \sqsubseteq N_X(y')$ or $N_X(y') \sqsubseteq N_X(y)$ for almost all $y' \in F$, and
- $\deg^X(y) = 25\varphi_F(y)$,

and note that $|F \setminus F'| = 0$. Fix $y \in F'$, and let $N = N_X(y)$ and $d = \deg^X(y) = |N| = f_F(\varphi_F(y))$. For almost every $y' \in F$ with $\varphi_F(y') > \varphi_F(y)$, we have that $\deg^X(y') = 25\varphi_F(y') > d$ and $N_X(y') \sqsupseteq N_X(y)$, which implies that $W(x, y') = 1$ for almost every $x \in N$. Since the measure of points in F with $\varphi_F(y') > \varphi_F(y)$

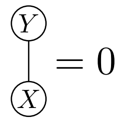

Figure 7: Decorated constraints forcing the zero tiles; (X,Y) is one of the pairs (C,D_D), (C,D_E), (C,D_F), (C,F), (E,D_A), (E,D_B), (E,D_C), (E,D_D), (E,D_E), (E,D_F), (E,E), (E,F) and (R,R).

is $\frac{1-d}{|F|}$, this implies that for almost every $x \in N$, we have $d_F(x) \geq 1-d$. An analogous argument implies that for almost every $x \in X \setminus N$, $d_F(x) \leq 1-d$. Since $f_X(\varphi_X(x)) = \deg_F(x)$ for almost every $x \in X$, we have obtained that for almost all $(x,y) \in X \times F$, $W(x,y) = 1$ iff $f_X(\varphi_X(x)) + f_F(\varphi_F(y)) \geq 1$. It follows that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in X \times F$ and that $f_X(z) = 25z$ for almost every $z \in [0, 1/25)$. In particular, it holds that $\eta_X(X) = f_X(\varphi_X(x))$ for $x \in X$ and $X \in \{A, B, D_A, \ldots, D_G, F\}$, i.e., we can think of the relative degree $\deg^F(x) = \eta_X(x)$ as the coordinate of the vertex $x \in X$.

6.3 Proof of Theorem 21 – black box

We now inspect the proof of Theorem 2 presented in [16]; the graphon W_0 from the statement of the theorem is depicted in Figure 1. The proof starts by presenting constraints that introduce the partition and the coordinate system analogously to Subsections 6.1 and 6.2. In particular, the relative degrees of vertices in the graphon W_0 from Theorem 2 with respect to the part P play the role of the coordinates that we have introduced in Subsection 6.2. The proof of Theorem 2 then continues by forcing the structure of the tiles $X \times Y$, $X, Y \in \{A, \ldots, G\}$, of the graphon W_0. This is done in Sections 3–5 of [16] with the empty tiles $B \times G$, $C \times G$ and $D \times G$ handled at the beginning of Section 6. We now include all decorated constraints depicted in Figures 5, 6, 8–11, 13, 15, 16, 18, 20 and 21 in [16], and the first constraint in Figure 22 from [16] to the family $\mathcal{C}$ with the decoration X, $X \in \{A, \ldots, G\}$, replaced with D_X, and the decoration P replaced with F. The arguments presented in Sections 3–5 of [16] apply in the same way in our setting. In particular, we get that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in X \times Y$ and all $X, Y \in \{D_A, \ldots, D_G\}$.

6.4 Proof of Theorem 21 – auxiliary tiles

In this section, we present the constraints forcing most of the tiles whose structure does not depend on either P or z. We start by forcing the density of the tiles that are zero to be zero, which is enforced by the constraints in Figure 7.

Consider the decorated constraints depicted in Figure 8. The first constraint implies that the following two properties hold for almost every $x \in A$:

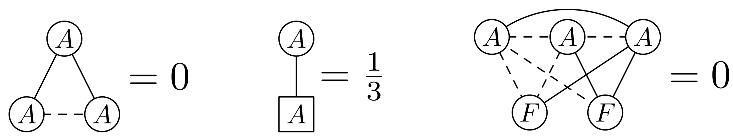

Figure 8: Decorated constraints forcing the structure of the tile $A \times A$.

- $W(y,z) = 1$ for almost any pair $(y,z) \in N_W^A(x) \times N_W^A(x)$,
- $W(y,z) = 0$ for almost any pair $(y,z) \in N_W^A(x) \times N_{1-W}^A(x)$.

Let Z be the set of $x \in A$ that have these two properties. In particular, the sets $N_W^A(x)$ and $N_{1-W}^A(x)$ intersect in a set of measure zero for every $x \in Z$, which implies that $W(x,y)$ is equal to 0 or 1 for every $x \in Z$ and almost every $y \in A$. Furthermore, for any $x, x' \in Z$, the sets $N_W^A(x)$ and $N_W^A(x')$ either differ on a set of measure zero, or their intersection has measure zero. Also note that given $x \in Z$, the set of $x' \in Z$ such that $N_W^A(x)$ and $N_W^A(x')$ differ on a set of measure zero is a measurable set.

Since having the same neighborhood up to measure zero is an equivalence relation, this implies that we can partition Z into a collection $\mathcal{J}$ of disjoint measurable subsets so that if x and x' belong to the same set, then their neighborhoods differ on a set of measure zero, and if they belong to different sets in $\mathcal{J}$, then their neighborhoods intersect in a set of measure zero. Furthermore, the conditions imply that for $x \in Z$, almost every $y \in N_W^Z(x)$ has the property that $N_W^Z(x)$ and $N_W^Z(y)$ differ on a set of measure zero. This implies that for almost every $x, y \in A$: $W(x,y) = 1$ if and only if there exists $J \in \mathcal{J}$ that contains both x and y. The second constraint implies that almost every $x \in A$ belongs to some $J \in \mathcal{J}$ and the measure of J is $|A|/3$. Hence, $\mathcal{J}$ contains exactly three sets and each has measure $|A|/3$.

Finally, the last constraint implies that for each $J \in \mathcal{J}$ and for almost all $x \in A$, if the measure of $x' \in J$ with $\eta_A(x') < \eta_A(x)$ is positive and the measure of $x'' \in J$ with $\eta_A(x'') > \eta_A(x)$ is positive, then $x \in J$. This implies that each $J \in \mathcal{J}$ differs from a preimage of an interval under η_A in a set of measure zero. Since each set in $\mathcal{J}$ has measure $1/3$, the three sets contained in $\mathcal{J}$ differ from $\eta_A^{-1}([0,1/3))$, $\eta_A^{-1}([1/3,2/3))$ and $\eta_A^{-1}([2/3,1))$ on a set of measure zero. It follows that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in A \times A$.

Fix $X \in \{B, C, D_A, \ldots, D_G, E\}$ and consider the decorated constraints depicted in Figure 9. The first constraint implies that almost every $x \in X$ has a neighborhood that is almost entirely contained in $\eta_A^{-1}([0,1/3))$, $\eta_A^{-1}([1/3,2/3))$, or $\eta_A^{-1}([2/3,1))$. This implies that there exist disjoint measurable subsets J_1, J_2 and J_3 of X such that $N^X(x) \sqsubseteq J_1$ for almost every $x \in \eta_A^{-1}([0,1/3))$, $N^X(x) \sqsubseteq J_2$ for almost every $x \in \eta_A^{-1}([1/3,2/3))$ and $N^X(x) \sqsubseteq J_3$ for almost every $x \in \eta_A^{-1}([2/3,1))$. The second constraint implies that $\deg^X(x) = 1/3$ for

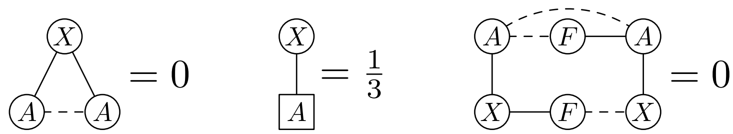

Figure 9: Decorated constraints forcing the structure of the tiles $A \times X$, where $X \in \{B, C, D_A, \ldots, D_G, E\}$.

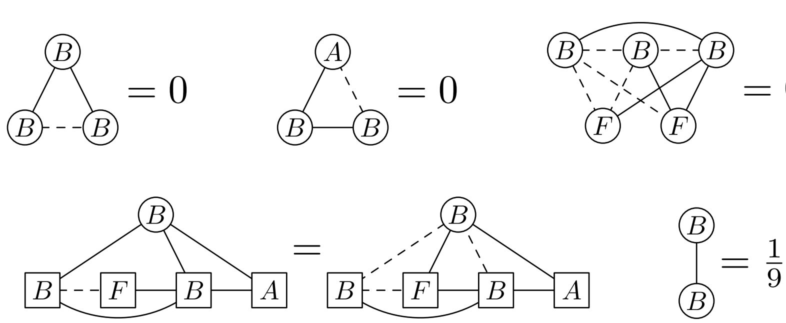

Figure 10: Decorated constraints forcing the structure of the tile $B \times B$.

almost every $x \in A$, which implies that $|J_1| = |J_2| = |J_3| = 1/3$ and, up to a set of measure zero, $W(x,y) = 1$ for $(x,y) \in A \times X$ if and only if either $\eta_A(x) \in [0,1/3)$ and $y \in J_1$, or $\eta_A(x) \in [1/3,2/3)$ and $y \in J_2$, or $\eta_A(x) \in [2/3,1)$ and $y \in J_3$. The last constraint trivially holds if $X = C$ or $X = E$, since in this case the tile $X \times F$ is 0 almost everywhere. If $X \notin \{C, E\}$, the constraint implies that $\eta_X(y_1) \le \eta_X(y_2) \le \eta_X(y_3)$ for almost any $y_1 \in J_1$, $y_2 \in J_2$ and $y_3 \in J_3$. Hence, we can assume that $J_1 = \eta_X^{-1}([0,1/3))$, $J_2 = \eta_X^{-1}([1/3,2/3))$ and $J_3 = \eta_X^{-1}([2/3,1))$. We conclude that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in A \times X$, where $X \in \{B, D_A, \ldots, D_G\}$.

If $X \in \{C, E\}$, we argue as follows. For $x \in X$, the first integral in (4) and (5) can have a value of $1/450$, $3/450$ or $5/450$. The differences between these values after multiplying by 900 or 4500, respectively, are larger than the maximum possible variation of the rest of the expressions in (4) and (5) (it is easy to see that they vary within an interval of length 1 and 7, respectively). It follows that $J_1 = \eta_X^{-1}([0,1/3))$, $J_2 = \eta_X^{-1}([1/3,2/3))$ and $J_3 = \eta_X^{-1}([2/3,1))$, which implies that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in A \times X$, $X \in \{C, E\}$.

We now turn our attention to the tile $B \times B$ and consider the decorated constraints depicted in Figure 10. We start with the three constraints on the first line. Following the arguments presented for the first constraint in Figure 8, we conclude that there exists a collection $\mathcal{J}$ of disjoint measurable subsets of B with positive measure such that for almost every $x, y \in B$, $W(x,y) = 1$ if and only if there exists $J \in \mathcal{J}$ that contains both x and y. The second constraint implies that $J \sqsubseteq \eta_B^{-1}([0,1/3))$, $J \sqsubseteq \eta_B^{-1}([1/3,2/3))$, or $J \sqsubseteq \eta_B^{-1}([2/3,1))$ for each

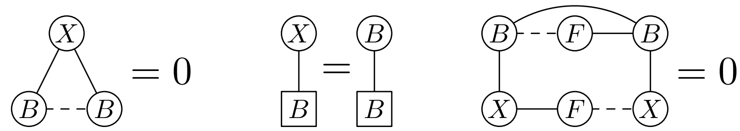

Figure 11: Decorated constraints forcing the structure of the tiles $B \times X$, where $X \in \{C, D_A, \ldots, D_G, E\}$.

$J \in \mathcal{J}$. Finally, the third constraint implies that for each $J \in \mathcal{J}$, there exists an interval $J' \subseteq [0, 1)$ such that J and $\eta_B^{-1}(J')$ differ on a set of measure zero (by the same argument as for the tile $A \times A$). Let $\mathcal{J}'$ be the set of such intervals J' for each $J \in \mathcal{J}$. Note that each interval in $\mathcal{J}'$ is a subinterval of $[0, 1/3)$, $[1/3, 2/3)$ or $[2/3, 1)$. The first constraint on the second line implies that if $J' = [a, b)$ is a subinterval of $[(i-1)/3, i/3)$, $i \in [3]$, then

$$b - a = i/3 - b\,.$$

Since the density of the tile $B \times B$ is equal to the sum of $|J'|^2$ for $J' \in \mathcal{J}'$, the density of $B \times B$ is at most $1/9$, with equality if and only if each $J' \in \mathcal{J}'$ is of the form $[(i - 2^{-(j-1)})/3, (i - 2^{-j})/3)$ for some $i \in [3]$ and $j \in \mathbb{N}$. We conclude from the last constraint that $W(x, y) = W_P(g(x), g(y))$ for almost all $(x, y) \in B \times B$.

Fix $X \in \{C, D_A, \ldots, D_G, E\}$ and consider the decorated constraints depicted in Figure 11. Let $\mathcal{J}'$ be the set containing the intervals from the analysis of the tile $B \times B$. Following the arguments for Figure 9, the first constraint implies that for each $J' \in \mathcal{J}'$, there exists a measurable subset J such that $N^X(x) \sqsubseteq J$ for each $x \in \eta_B^{-1}(J')$. Furthermore, the subsets J are disjoint for different $J' \in \mathcal{J}'$. The second constraint implies that $\deg^X(x) = \deg^B(x) = |J'|$ for almost every $x \in \eta_B^{-1}(J')$, which implies that $|J| = |J'| \cdot |X|$ for every $J' \in \mathcal{J}'$ (since the sum of the measures of the intervals in $\mathcal{J}'$ is one), and $N^X(x) \sqsupseteq J$. If $X \in \{D_A, \ldots, D_G\}$, the last constraint implies that each J is a preimage of an interval and these intervals follow the order of the intervals in $\mathcal{J}'$ (note that as before, this constraint trivially holds if $X = C$ or $X = E$). It follows that J and $\eta_X^{-1}(J')$ differ on a set of measure zero. We conclude that $W(x, y) = W_P(g(x), g(y))$ for almost all $(x, y) \in B \times X$, where $X \in \{D_A, \ldots, D_G\}$. If $X = C$, then (4) implies that J is a preimage of an interval and these intervals follow the order of the intervals in $\mathcal{J}'$, which again leads to the conclusion that $W(x, y) = W_P(g(x), g(y))$ for almost all $(x, y) \in B \times C$. The analysis of the tile $B \times E$ will be finished in Subsection 6.5.

We now analyze the tile $C \times D_A$. Consider the decorated constraints depicted in Figure 12. Let a_1, a_2 and a_3 be the three A-roots from the constraints on the first line in the figure. Almost any choice of the roots satisfies that $\eta_A(a_1) < \eta_A(a_2) < \eta_A(a_3)$. Since the three roots are non-adjacent, it follows that almost any choice of them satisfies that $\eta_A(a_i) \in [(i-1)/3, i/3)$, $i \in [3]$, because of the structure of the tile $A \times A$. For the first constraint, the structure of the tile

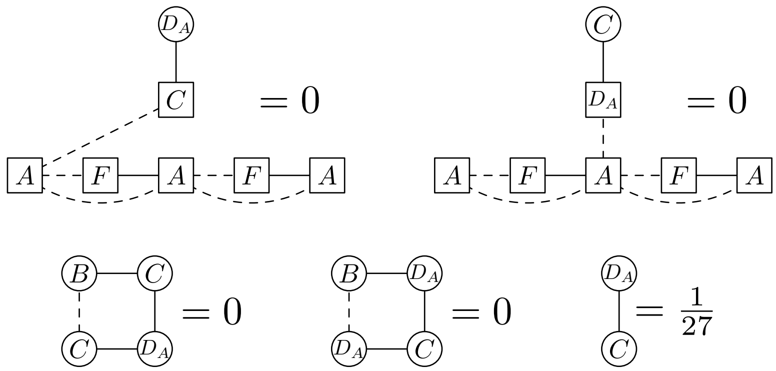

Figure 12: Decorated constraints forcing the structure of the tile $C \times D_A$.

$A\times C$ implies that almost any choice of the C-root x satisfies that $\eta_C(x) \in [1/3, 1)$. Likewise, the structure of the tile $A\times D_A$ implies that almost any choice of the D_A-root y in the second constraint satisfies that $\eta_{D_A}(y) \in [0, 1/3)\cup[2/3, 1)$. It follows that $W(x, y) = 0$ for almost all $(x, y) \in (C\times D_A)\setminus(\eta_C^{-1}([0, 1/3))\times\eta_{D_A}^{-1}([1/3, 2/3)))$.

We now focus on the constraints on the second line in Figure 12. The first constraint implies that the neighborhood of almost every $y \in D_A$ is contained in the set

$$\{x \in C \text{ such that } \lfloor 3\eta_C(x)\rfloor = j \text{ and } \llbracket 3\eta_C(x)\rrbracket = k\}$$

for some integers j and k. Note that by the previous paragraph we must have $j = 0$. This implies that there exist disjoint measurable subsets $J_i \subseteq D_A$, $i \in \mathbb{N}$, such that $N^{D_A}(x) \sqsubseteq J_{\llbracket 3\eta_C(x)\rrbracket}$ for almost every $x \in C$. Likewise, the second constraint implies that there exist disjoint measurable subsets $J'_i \subseteq C$, $i \in \mathbb{N}$, such that $N^C(y) \sqsubseteq J'_{\llbracket 3\eta_{D_A}(y)\rrbracket}$ for almost every $y \in D_A$. Hence, there exists a function $f : \mathbb{N} \to \mathbb{N}_0$ that is injective on $f^{-1}(\mathbb{N})$ such that the following holds for almost every $(x, y) \in C \times D_A$: $W(x, y) > 0$ only if $f(\llbracket 3\eta_C(x)\rrbracket) = \llbracket 3\eta_{D_A}(y)\rrbracket$. However, the last constraint on the second line can hold only if $f(i) = i$ for all $i \in \mathbb{N}$ (otherwise, the value would be strictly less than $1/27$). It follows that $W(x, y) = W_P(g(x), g(y))$ for almost all $(x, y) \in C\times D_A$. In a very analogous way, the constraints depicted in Figure 13 guarantee that $W(x, y) = W_P(g(x), g(y))$ for almost all $(x, y) \in C \times D_B$.

We now consider the decorated constraints depicted in Figure 14 and analyze the tile $C \times D_C$. Let b_1 and b_2 be the two B-roots in the first constraint. The choices of the roots vary through almost all pairs $(b_1, b_2) \in B\times B$ such that either $\lfloor 3\eta_B(b_1)\rfloor < \lfloor 3\eta_B(b_2)\rfloor$, or $\lfloor 3\eta_B(b_1)\rfloor = \lfloor 3\eta_B(b_2)\rfloor$ and $\llbracket 3\eta_B(b_1)\rrbracket < \llbracket 3\eta_B(b_2)\rrbracket$. Hence, the first constraint implies that $\deg^{D_C}(x) = 0$ for any $x \in C$ with $\eta_C(x) \in [1/6, 1)$. Similarly, the second constraint implies that $\deg^C(y) = 0$ for any $y \in D_C$ with $\eta_{D_C}(y) \in [1/6, 1)$. Since the density of the tile $C \times D_C$ is $1/36$ by the third constraint, it follows that $W(x, y) = 1$ for almost every $(x, y) \in \eta_C^{-1}([0, 1/6)) \times$

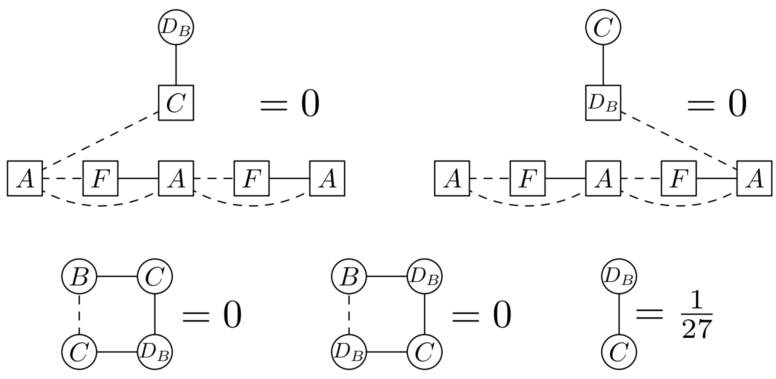

Figure 13: Decorated constraints forcing the structure of the tile $C \times D_B$.

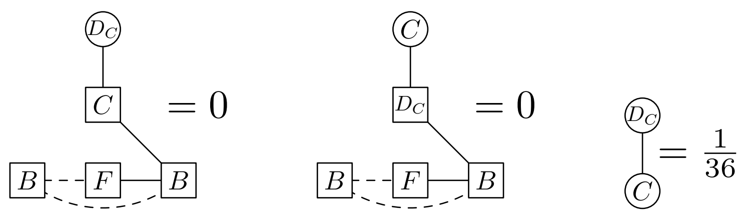

Figure 14: Decorated constraints forcing the structure of the tile $C \times D_C$.

$\eta_{D_C}^{-1}([0,1/6))$, and $W(x,y)=0$ for almost every $(x,y)\in(C\times D_C)\setminus(\eta_C^{-1}([0,1/6))\times\eta_{D_C}^{-1}([0,1/6)))$. We conclude that $W(x,y)=W_P(g(x),g(y))$ for almost all $(x,y)\in C\times D_C$.

In the subsequent sections, we finish the proof by forcing the structure of the tiles $C\times C$, $C\times D_G$, $C\times E$ and $D_G\times E$, forcing all tiles involving parts Q and R, and finishing the analysis of the tile $B\times E$.

6.5 Proof of Theorem 21 – bounding polynomials

We next force some general structure of the tiles $C\times C$, $C\times E$ and $D_G\times E$. Consider the decorated constraints depicted in Figure 15. In all four constraints, almost every choice of the A-roots a_1, a_2 and a_3 satisfies that $\eta_A(a_1)\in[0,1/3)$, $\eta_A(a_2)\in[1/3,2/3)$ and $\eta_A(a_3)\in[2/3,1)$. The first constraint then implies that $\deg^C(x)=0$ for almost every $x\in\eta_C^{-1}([2/3,1))$. In the second constraint, almost any pair of non-adjacent B-vertices b_1 and b_2 adjacent to $a_1\in\eta_A^{-1}([0,1/3))$ satisfies that $[\![3\eta_B(b_1)]\!]\neq[\![3\eta_B(b_2)]\!]$. It follows that $W(x,y)=0$ for almost every $(x,y)\in\eta_C^{-1}([0,1/3))\times\eta_C^{-1}([0,1/3))$ such that $[\![3\eta_C(x)]\!]\neq[\![3\eta_C(y)]\!]$. Similarly, the first constraint on the second line yields that $W(x,y)=0$ for almost every $(x,y)\in\eta_C^{-1}([1/3,2/3))\times\eta_C^{-1}([1/3,2/3))$ such that $[\![3\eta_C(x)]\!]\neq[\![3\eta_C(y)]\!]$. Finally, in the last constraint, almost any neighbor $z\in D_A$ of b_1 satisfies that $\eta_{D_A}(z)\in$

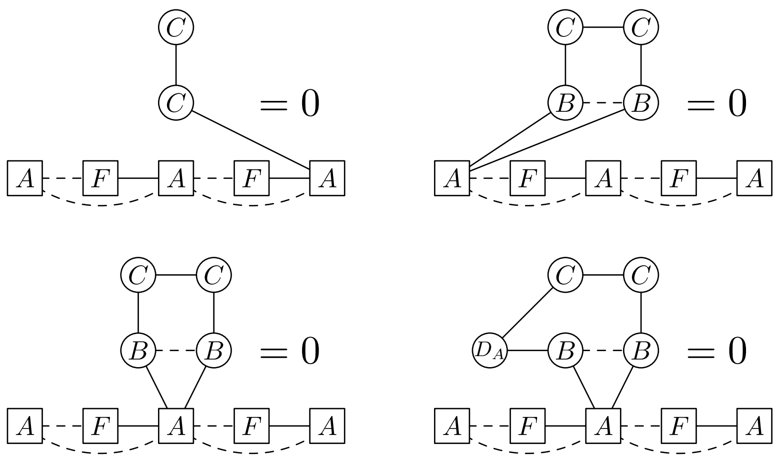

Figure 15: Decorated constraints forcing the general structure of the tile $C \times C$.

$[1/3, 2/3)$ and $[\![3\eta_{D_A}(z)]\!] = [\![3\eta_B(b_1)]\!]$. It follows that $W(x,y) = 0$ for almost every $(x,y) \in \eta_C^{-1}([0,1/3)) \times \eta_C^{-1}([1/3,2/3))$ such that $[\![3\eta_C(x)]\!] \neq [\![3\eta_C(y)]\!]$. We conclude that $W(x,y) = 0$ for almost every pair $(x,y) \in C \times C$, with the possible exception of those (x,y) such that $\eta_C(x) \in [0,2/3)$, $\eta_C(y) \in [0,2/3)$, and $[\![3\eta_C(x)]\!] = [\![3\eta_C(y)]\!]$.

In Subsection 6.4, we showed that there exist disjoint measurable subsets $J_{i,j} \subseteq E$, $i \in [3]$ and $j \in \mathbb{N}$, with $|J_{i,j}| = 2^{-j}/3$ such that the following holds for almost every $x \in B$: $W(x,y) = 1$ for almost every $y \in J_{\lfloor 3\eta_B(x)\rfloor+1,[\![3\eta_B(x)]\!]}$ and $W(x,y) = 0$ for almost every $y \notin J_{\lfloor 3\eta_B(x)\rfloor+1,[\![3\eta_B(x)]\!]}$. Following the lines of the analysis of the constraints in Figure 15 with respect to the tile $C \times C$, we conclude that the decorated constraints depicted in Figure 16 imply that $W(x,y) = 0$ for almost every pair $(x,y) \in C \times E$ that does not satisfy that $\eta_C(x) \in [0,2/3)$ and $y \in J_{1,[\![3\eta_C(x)]\!]} \cup J_{2,[\![3\eta_C(x)]\!]}$. Similarly, the decorated constraints depicted in Figure 17 imply that $W(x,y) = 0$ for almost every pair $(x,y) \in D_G \times E$ that does not satisfy that $\eta_{D_G}(x) \in [1/3,1)$ and $y \in J_{1,[\![3\eta_{D_G}(x)]\!]}$.

We are now ready to finish the analysis of $B \times E$. The structure of the tiles $C \times E$ and $D_G \times E$ implies that for almost every $x \in E$, $\deg_W^C(x) \leq \deg_W^B(x)$ and $\deg_W^{D_G}(x) \leq 2\deg_W^B(x)$. This implies that

$$5\deg_W^B(x) + \deg_W^{D_G}(x) - \int_{N_W^C(x)} \deg_W^{D_A}(z)\,\mathrm{d}z \in [4\deg_W^B(x), 7\deg_W^B(x)]$$

for almost every $x \in E$. In particular, for almost all $x \in J_{i,j}$ and $x' \in J_{i,j'}$ with $i \in [3]$ and $j < j'$, the intervals $[4\deg_W^B(x), 7\deg_W^B(x)]$ and $[4\deg_W^B(x'), 7\deg_W^B(x')]$ are disjoint, since $\deg_W^B(x') \geq 2\deg_W^B(x)$. This implies that $\eta_E^{-1}(J_{i,j})$ and $[(i - 2^{-j-1})/3, (i-2^{-j})/3)$ differ on a set of measure zero for all $i \in [3]$ and $j \in \mathbb{N}$. Consequently, $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in B \times E$.

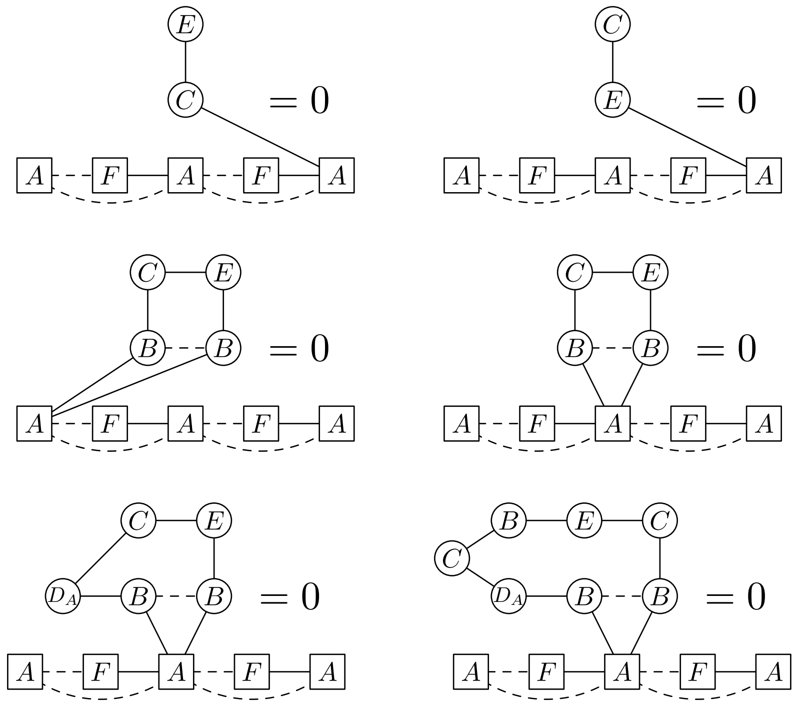

Figure 16: Decorated constraints forcing the general structure of the tile $C \times E$.

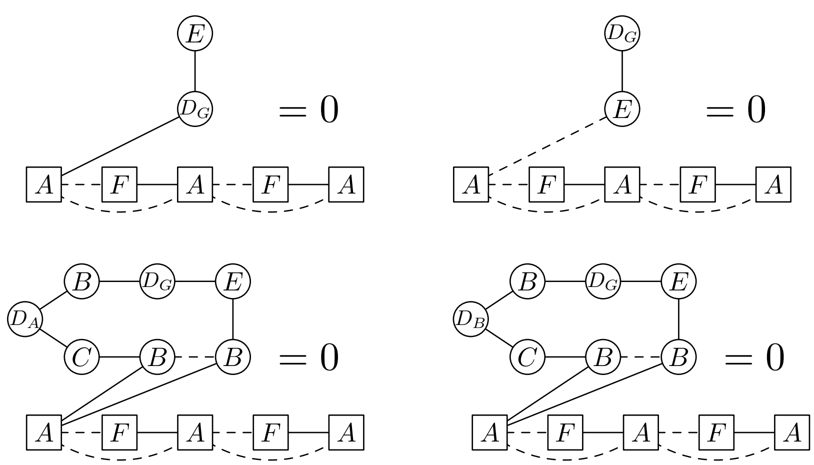

Figure 17: Decorated constraints forcing the general structure of the tile $D_G \times E$.

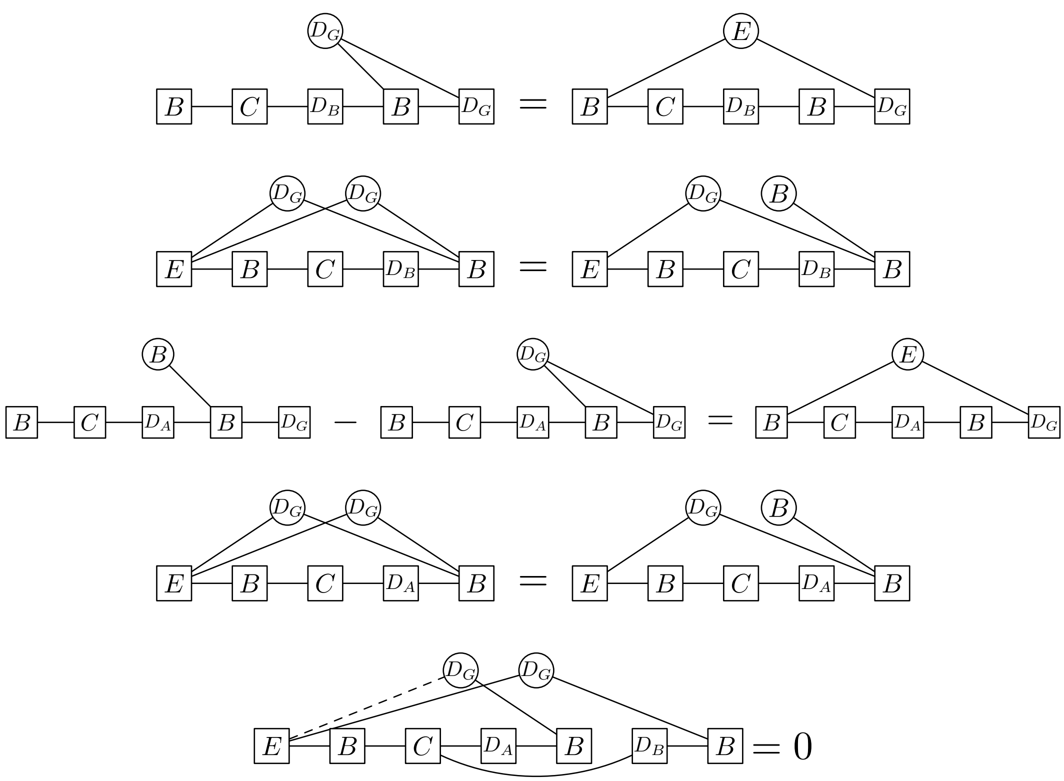

Figure 18: Decorated constraints forcing the specific structure of the tile $D_G \times E$.

We now analyze the decorated constraints depicted in Figure 18. In the first constraint, almost every choice of the B-roots b_1 and b_2 and the D_G-root d satisfies that $\eta_B(b_1) \in [0,1/3)$, $\eta_B(b_2) \in [2/3,1)$, $\eta_{D_G}(d) \in [2/3,1)$, and $[\![3\eta_B(b_1)]\!] = [\![3\eta_B(b_2)]\!] = [\![3\eta_{D_G}(d)]\!]$; fix such a choice of roots and let i be the common value of $[\![3\eta_B(b_1)]\!] = [\![3\eta_B(b_2)]\!] = [\![3\eta_{D_G}(d)]\!]$. Together with the structure of $D_G \times D_G$, the constraint implies that $\deg^E(x) = l_i \cdot 2^{-i}/3$ for almost every $x \in \eta_{D_G}^{-1}([2/3,1))$ where $i = [\![3\eta_{D_G}(d)]\!]$ (the value l_i is given by the bounding sequence P).

In the second constraint, almost every choice of the E-root e and the two B-roots b_1 and b_2 satisfies that $\eta_E(e) \in [0,1/3)$, $\eta_B(b_1) \in [0,1/3)$, $\eta_B(b_2) \in [2/3,1)$, and $[\![3\eta_E(e)]\!] = [\![3\eta_B(b_1)]\!] = [\![3\eta_B(b_2)]\!]$. Let α be the relative degree of e with respect to $\eta_{D_G}^{-1}([2/3,1))$. The constraint implies that $\alpha^2 = \alpha 2^{-i}$, where $i = [\![3\eta_B(b_2)]\!]$. Therefore, the constraint is satisfied if and only if $\alpha \in \{0, 2^{-i}\}$. Hence, almost every vertex $x \in \eta_E^{-1}([0,1/3))$ satisfies that either $W(x,y) = 1$ for almost every $y \in \eta_{D_G}^{-1}([2/3,1))$ with $[\![3\eta_E(x)]\!] = [\![3\eta_{D_G}(y)]\!]$, or $W(x,y) = 0$ for almost every $y \in \eta_{D_G}^{-1}([2/3,1))$ with $[\![3\eta_E(x)]\!] = [\![3\eta_{D_G}(y)]\!]$. It follows that there exist $J_{1,i}^l \subseteq J_{1,i}$, $i \in \mathbb{N}$, such that $|J_{1,i}^l| = l_i \cdot |J_{1,i}|$ and for almost every $x \in \eta_E^{-1}([0,1/3))$ and $y \in \eta_{D_G}^{-1}([2/3,1))$ it holds that $W(x,y) = 1$ if and only if $x \in J^l_{1,[\![3\eta_{D_G}(y)]\!]}$, and $W(x,y) = 0$ otherwise.

An analogous argument applies with respect to the third and the fourth constraints; note that the values on the densities in the diagonal squares in the $\eta_{D_G}^{-1}([1/3,2/3)) \times \eta_{D_G}^{-1}([1/3,2/3))$ tiles are $1 - u_i$. Hence, the third and the fourth constraints imply the existence of $J_{1,i}^u \subseteq J_{1,i}$, $i \in \mathbb{N}$, such that $|J_{1,i}^u| = u_i \cdot |J_{1,i}|$ and for almost every $x \in \eta_E^{-1}([0,1/3))$ and $y \in \eta_{D_G}^{-1}([1/3,2/3))$ it holds $W(x,y) = 1$ if and only if $x \in J^u_{1,[\![3\eta_{D_G}(y)]\!]}$ and $W(x,y) = 0$ otherwise.

We next consider the final constraint in Figure 18. Almost every choice of the E-root e and the B-roots b_1, b_2 and b_3 satisfies that $\eta_E(e) \in [0,1/3)$, $\eta_B(b_i) \in [(i-1)/3, i/3)$, $i \in \{1,2,3\}$, and $[\![3\eta_E(e)]\!] = [\![3\eta_B(b_1)]\!] = [\![3\eta_B(b_2)]\!] = [\![3\eta_B(b_3)]\!]$. Hence, the constraint is satisfied if and only if $J_{1,i}^l \sqsubseteq J_{1,i}^u$ for all $i \in \mathbb{N}$. We now observe that the third term in (5) for almost every $x \in E$ is $2^{-i+1}/3$ (if $x \in J_{1,i}^l$), $2^{-i}/3$ (if $x \in J_{1,i}^u \setminus J_{1,i}^l$) or 0 (if $x \notin J_{1,i}^u$) where $i = [\![3\eta_E(x)]\!]$. In particular, it dominates the last term in (5), which cannot exceed $2^{-2i+2}/9$. It follows that for every $i \in \mathbb{N}$ it holds $\eta_E(x) \le \eta_E(x') \le \eta_E(x'')$ for almost all $x \in J_{1,i}^l$, $x' \in J_{1,i}^u \setminus J_{1,i}^l$ and $x'' \in J_{1,i} \setminus J_{1,i}^u$. Consequently, it holds that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in E \times D_G$.

We now consider the three decorated constraints depicted in Figure 19. For $(i,j) \in \{1,2,3\} \times \mathbb{N}$, let

$$I_{i,j} = \left[\frac{i - 2^{1-j}}{3}, \frac{i - 2^{-j}}{3}\right).$$

We start with the first constraint. Let b_1 and b_2 be the two B-roots and assume that they belong to $\eta_B^{-1}(I_{i_1,j_1})$ and $\eta_B^{-1}(I_{i_2,j_2})$, respectively, $(i_1,j_1),(i_2,j_2) \in \{1,2,3\} \times \mathbb{N}$. For almost every choice of the roots b_1 and b_2, almost all the

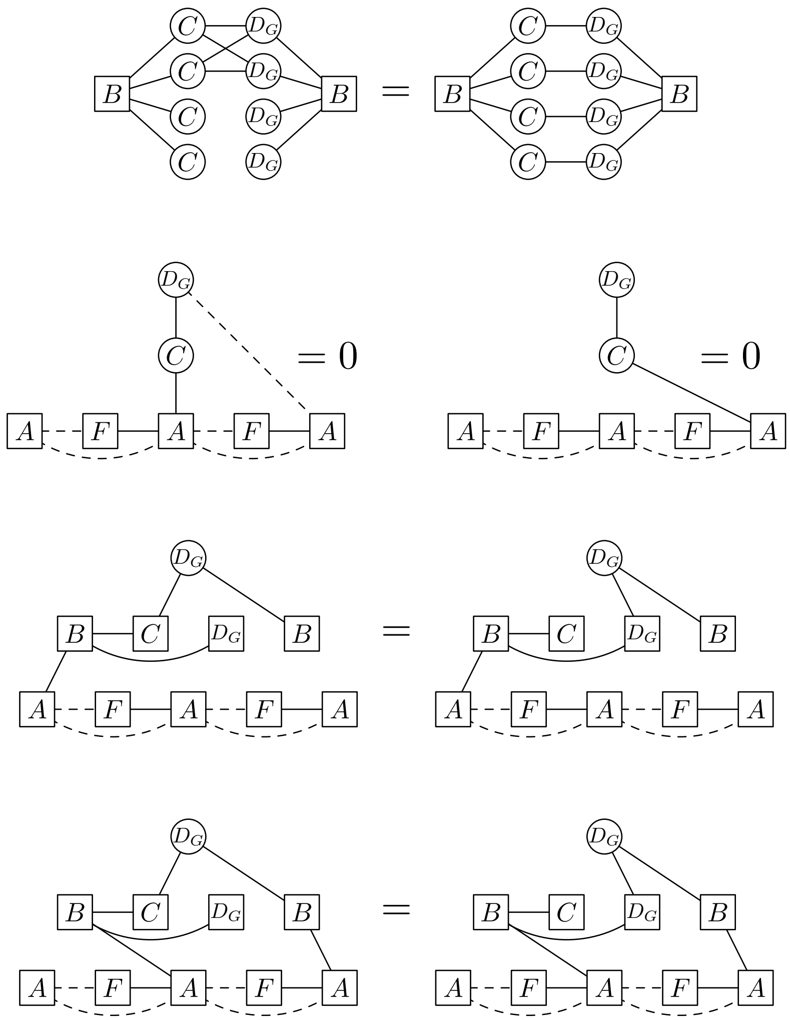

Figure 19: Decorated constraints forcing the structure of the tile $C \times D_G$.

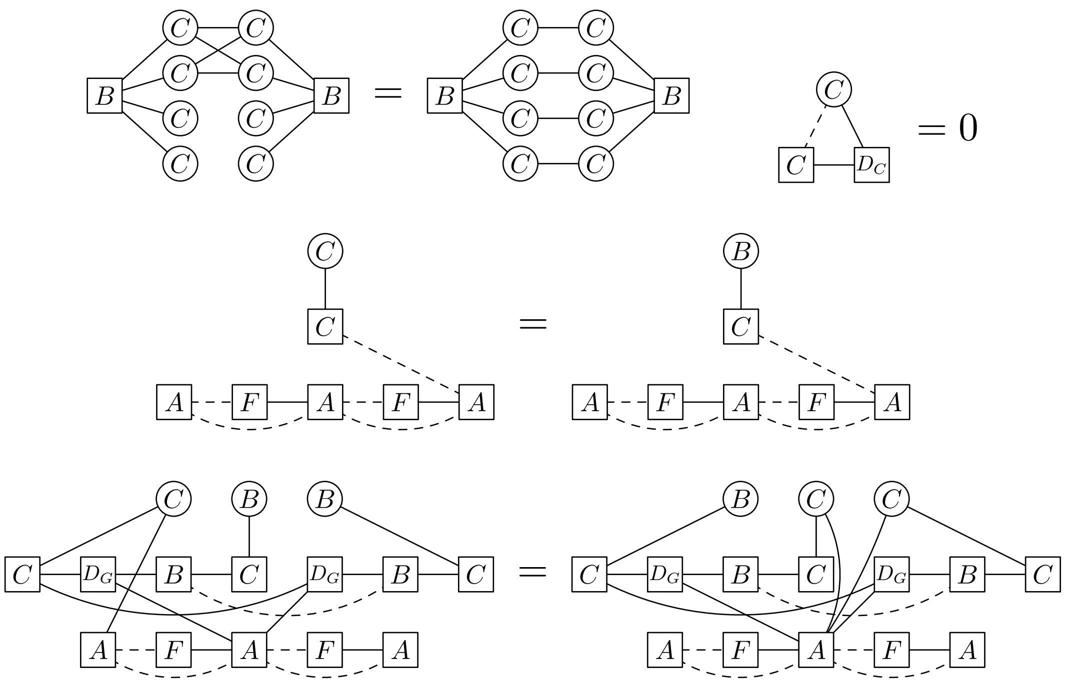

Figure 20: Decorated constraints forcing the specific structure of the tile $C \times C$.

choices of the C-vertices belong to $\eta_C^{-1}(I_{i_1,j_1})$ and almost all the choices of the D_G-vertices to $\eta_{D_G}^{-1}(I_{i_2,j_2})$. This implies that the density of 4-cycles in the tile $C \times D_G$ restricted to $I_{i_1,j_1} \times I_{i_2,j_2}$ is equal to the fourth power of the density of the tile restricted to $I_{i_1,j_1} \times I_{i_2,j_2}$. This implies (see for example the proof of Claim 11.63 in [32]) that W is equal to a constant ξ_{i_1,j_1,i_2,j_2} almost everywhere on each rectangle $\eta_C^{-1}(I_{i_1,j_1}) \times \eta_{D_G}^{-1}(I_{i_2,j_2})$ for each $(i_1,j_1),(i_2,j_2) \in \{1,2,3\} \times \mathbb{N}$. The two constraints on the second line imply that $\xi_{i_1,j_1,i_2,j_2} = 0$ for $(i_1,i_2) \in \{(2,1),(2,2),(3,1),(3,2),(3,3)\}$. The constraint on the third line in Figure 19 implies that the values ξ_{i_1,j_1,i_2,j_2} are equal to the corresponding values inside the tile $D_G \times D_G$ for $(i_1,i_2) \in \{(1,1),(1,2),(1,3)\}$, and the constraint on the fourth line implies the same for $(i_1,i_2) = (2,3)$. It now follows that $W(x,y) = W_P(g(x),g(y))$ for almost all $(x,y) \in C \times D_G$.

6.6 Proof of Theorem 21 – variable tiles

In the previous subsection, we showed that $W(x,y) = 0$ for almost every $(x,y) \in \eta_C^{-1}([0,1/3)) \times \eta_C^{-1}([1/3,2/3))$ such that $[\![3\eta_C(x)]\!] \neq [\![3\eta_C(y)]\!]$, and we also showed that $W(x,y) = 0$ for almost every $x \in \eta_C^{-1}([2/3,1))$ and $y \in C$. The first constraint in Figure 20 is analogous to the first constraint in Figure 19 and it implies that there exist α_i^{11}, α_i^{12} and α_i^{22}, $i \in \mathbb{N}$, such that the following holds for almost every $(x,y)\in C\times C$:

$$W(x,y)=\begin{cases}\alpha_i^{11} & \text{if } \lfloor 3\eta_C(x)\rfloor=0,\ \lfloor 3\eta_C(y)\rfloor=0 \text{ and } i=[\![3\eta_C(x)]\!]=[\![3\eta_C(y)]\!],\\ \alpha_i^{12} & \text{if } \lfloor 3\eta_C(x)\rfloor=0,\ \lfloor 3\eta_C(y)\rfloor=1 \text{ and } i=[\![3\eta_C(x)]\!]=[\![3\eta_C(y)]\!],\\ \alpha_i^{12} & \text{if } \lfloor 3\eta_C(x)\rfloor=1,\ \lfloor 3\eta_C(y)\rfloor=0 \text{ and } i=[\![3\eta_C(x)]\!]=[\![3\eta_C(y)]\!],\\ \alpha_i^{22} & \text{if } \lfloor 3\eta_C(x)\rfloor=1,\ \lfloor 3\eta_C(y)\rfloor=1 \text{ and } i=[\![3\eta_C(x)]\!]=[\![3\eta_C(y)]\!],\\ 0 & \text{otherwise.}\end{cases}$$

The second constraint on the first line in Figure 20 yields that $\alpha_1^{11}=1$. The constraint on the second line in the figure implies that for every $i\in\mathbb{N}$

$$\frac{\alpha_i^{11}+\alpha_i^{12}}{3\cdot 2^i}=\frac{1}{3\cdot 2^i}\qquad\text{and}\qquad\frac{\alpha_i^{12}+\alpha_i^{22}}{3\cdot 2^i}=\frac{1}{3\cdot 2^i}.$$

It follows that $\alpha_i^{11}=\alpha_i^{22}=1-\alpha_i^{12}$. For $i\in\mathbb{N}$, set $z_i=\alpha_j^{11}$ for j with $M_j=\{i\}$.

We now analyze the last constraint depicted in Figure 20. For almost every choice of C-roots c_1, c_2 and c_3, the adjacencies to the middle A-root imply that $\eta_C(c_1)\in[0,1/3)$, $\eta_C(c_2)\in[1/3,2/3)$ and $\eta_C(c_3)\in[1/3,2/3)$. The structure of the tile $C\times D_G$ implies that $[\![3\eta_C(c_1)]\!]\geq 2$ and that $M_{[\![3\eta_C(c_2)]\!]}$ is either $\{\min M_{[\![3\eta_C(c_1)]\!]}\}$ or $M_{[\![3\eta_C(c_1)]\!]}\setminus\{\min M_{[\![3\eta_C(c_1)]\!]}\}$. Similarly, $M_{[\![3\eta_C(c_3)]\!]}$ is either $\{\min M_{[\![3\eta_C(c_1)]\!]}\}$ or $M_{[\![3\eta_C(c_1)]\!]}\setminus\{\min M_{[\![3\eta_C(c_1)]\!]}\}$. Since the two B-roots are not adjacent, it follows that $M_{[\![3\eta_C(c_2)]\!]}\neq M_{[\![3\eta_C(c_3)]\!]}$. By the symmetry of the graphs in the constraint, we can assume that $M_{[\![3\eta_C(c_2)]\!]}=\{\min M_{[\![3\eta_C(c_1)]\!]}\}$ and $M_{[\![3\eta_C(c_3)]\!]}=M_{[\![3\eta_C(c_1)]\!]}\setminus\{\min M_{[\![3\eta_C(c_1)]\!]}\}$ further in the analysis.

For such a choice of the C-roots c_1, c_2 and c_3, the constraint implies that

$$\frac{\alpha^{11}_{[\![3\eta_C(c_1)]\!]}}{27\cdot 2^{[\![3\eta_C(c_1)]\!]+[\![3\eta_C(c_2)]\!]+[\![3\eta_C(c_3)]\!]}}=\frac{\alpha^{22}_{[\![3\eta_C(c_2)]\!]}\alpha^{22}_{[\![3\eta_C(c_3)]\!]}}{27\cdot 2^{[\![3\eta_C(c_1)]\!]+[\![3\eta_C(c_2)]\!]+[\![3\eta_C(c_3)]\!]}}.$$

Since $\alpha_i^{11}=\alpha_i^{22}$ for each $i\in\mathbb{N}$, this yields that $\alpha^{11}_{[\![3\eta_C(c_1)]\!]}=\alpha^{11}_{[\![3\eta_C(c_2)]\!]}\alpha^{11}_{[\![3\eta_C(c_3)]\!]}$.

Let $k=[\![3\eta_C(c_1)]\!]$. If $|M_k|=1$, then $[\![3\eta_C(c_2)]\!]=k$ and $[\![3\eta_C(c_3)]\!]=1$. In this case, the constraint always holds. If $|M_k|>1$, then the constraint implies that $\alpha_k^{11}=\alpha_{k'}^{11}\cdot\alpha_{k''}^{11}$ where k' and k'' are such that $M_{k'}=\{\min M_k\}$ and $M_{k''}=M_k\setminus\{\min M_k\}$. By induction on the size of M_k, we get that $\alpha_k^{11}=z^{M_k}$ for every $k\in\mathbb{N}$. For our choice of z_i, $i\in\mathbb{N}$, it follows that $W(x,y)=W_P(g(x),g(y))$ for almost all $(x,y)\in C\times C$. Note that the values of z_i, $i\in\mathbb{N}$, were uniquely determined by the structure of the graphon W.

We now focus on the tile $C\times E$ and consider the constraints depicted in Figure 21. In Subsection 6.5, we showed that $W(x,y)=0$ for almost every $(x,y)\in C\times E$ except for those (x,y) such that $\eta_C(x)\in[0,2/3)$ and $y\in J_{1,[\![3\eta_C(x)]\!]}\cup J_{2,[\![3\eta_C(x)]\!]}$. Let us consider the first constraint depicted in the figure. For almost every choice of E-root y and right B-root x, both sides of the constraint are zero unless $\eta_B(x)\in[0,2/3)$ and $y\in J_{1,[\![3\eta_B(x)]\!]}\cup J_{2,[\![3\eta_B(x)]\!]}$. If $\eta_B(x)\in[0,1/3)$ and $y\in J_{1,[\![3\eta_B(x)]\!]}\cup J_{2,[\![3\eta_B(x)]\!]}$, then the constraint implies that

$$\left(\deg^{\eta_C^{-1}([0,1/3))}(y)\right)^2=2^{-[\![3\eta_B(x)]\!]}\deg^{\eta_C^{-1}([0,1/3))}(y)/3\,;$$

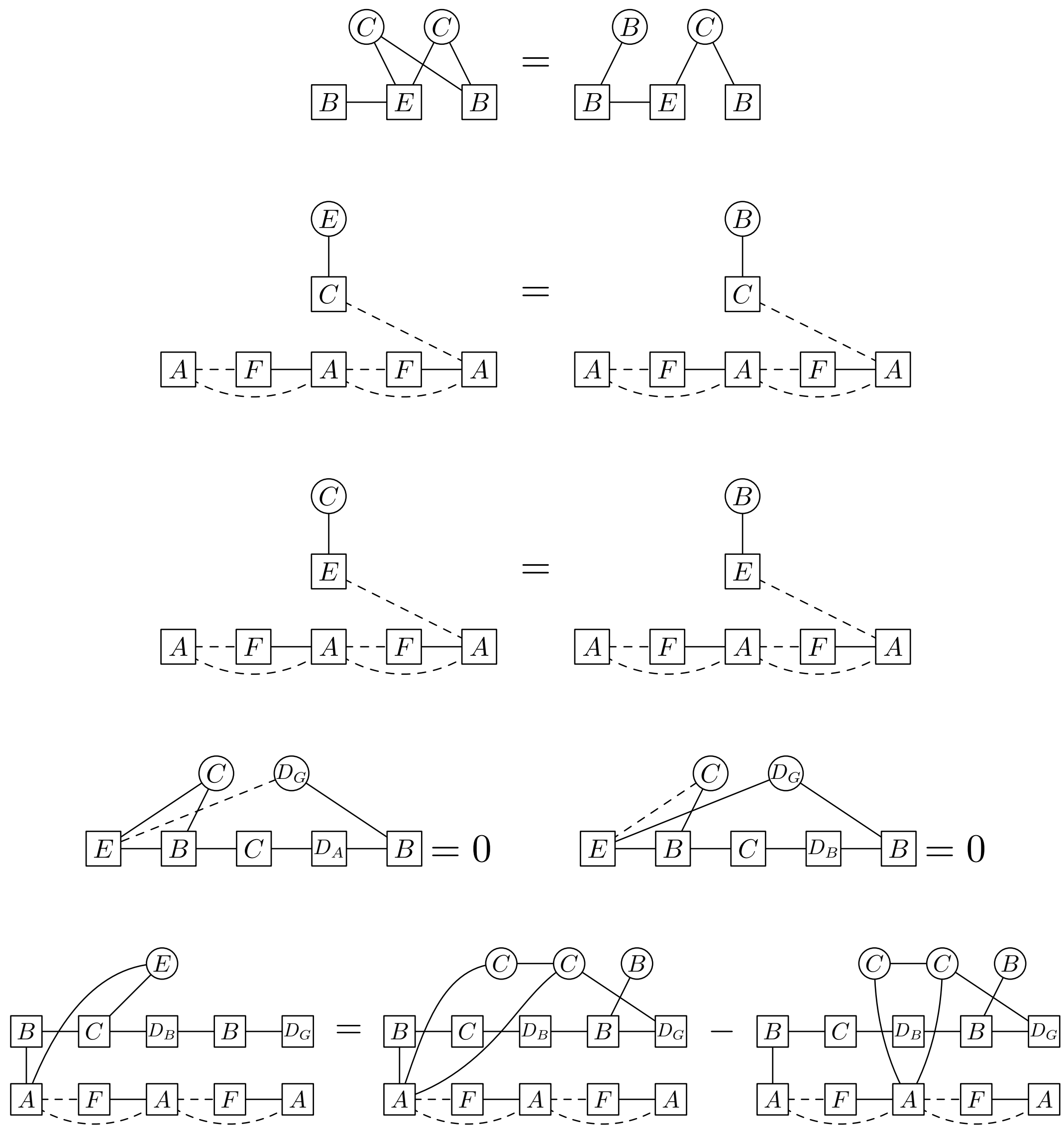

Figure 21: Decorated constraints forcing the specific structure of the tile $C \times E$.

if $\eta_B(x) \in [1/3, 2/3)$ and $y \in J_{1,[\![3\eta_B(x)]\!]} \cup J_{2,[\![3\eta_B(x)]\!]}$, then the constraint implies that

$$\left(\deg^{\eta_C^{-1}([1/3,2/3))}(y)\right)^2 = 2^{-[\![3\eta_B(x)]\!]} \deg^{\eta_C^{-1}([1/3,2/3))}(y)/3\,.$$

Hence, there exist $J_{11,i} \subseteq J_{1,i}$, $J_{12,i} \subseteq J_{1,i}$, $J_{21,i} \subseteq J_{2,i}$ and $J_{22,i} \subseteq J_{2,i}$ such that $W(x,y) = 1$ for almost every $(x,y) \in C \times E$ such that either $\eta_C(x) \in [0, 1/3)$ and $y \in J_{11,[\![3\eta_C(x)]\!]} \cup J_{21,[\![3\eta_C(x)]\!]}$ or $\eta_C(x) \in [1/3, 2/3)$ and $y \in J_{12,[\![3\eta_C(x)]\!]} \cup J_{22,[\![3\eta_C(x)]\!]}$, and $W(x,y) = 0$ for almost all other $(x,y) \in C \times E$.

The second constraint depicted in Figure 21 yields that $\deg^E(x) = \deg^B(x) = 2^{-[\![\eta_C(x)]\!]}/3$ for almost every $x \in C$ with $\eta_C(x) \in [0, 2/3)$. Hence, $|J_{11,i}| + |J_{21,i}| = 2^{-i}/3$ and $|J_{12,i}| + |J_{22,i}| = 2^{-i}/3$ for every $i \in \mathbb{N}$. The third constraint in Figure 21 yields that $\deg^C(y) = \deg^B(y) = 2^{-[\![\eta_E(y)]\!]}/3$ for almost every $y \in E$ with $\eta_E(y) \in [0, 2/3)$. This implies that we can assume that the sets $J_{11,i}$ and $J_{12,i}$ are disjoint, the sets $J_{21,i}$ and $J_{22,i}$ are disjoint, and $J_{11,i} \cup J_{12,i} = J_{1,i}$ and $J_{21,i} \cup J_{22,i} = J_{2,i}$ for every $i \in \mathbb{N}$.

The left constraint on the fourth line implies that almost every $y \in J_{11,i}$ satisfies that $y \in J^u_{1,i}$, $i \in \mathbb{N}$, and the right constraint implies that almost every $y \in J^l_{1,i}$ satisfies that $y \in J_{11,i}$, $i \in \mathbb{N}$ (the sets $J^l_{1,i}$ and $J^u_{1,i}$ were defined in Subsection 6.5). It follows that $J^l_{1,i} \sqsubseteq J_{11,i} \sqsubseteq J^u_{1,i}$ for all $i \in \mathbb{N}$, which in particular implies that $|J^l_{1,i}| \le |J_{11,i}| \le |J^u_{1,i}|$, i.e., $3 \cdot 2^i \cdot |J_{11,i}| \in [l_i, u_i]$. Because of the last term in (5), we get that $\eta_E(y) \le \eta_E(y')$ for almost all $y \in J_{11,i}$ and $y' \in J_{12,i}$. Similarly, we get that $\eta_E(y) \le \eta_E(y')$ for almost all $y \in J_{21,i}$ and $y' \in J_{22,i}$. To show that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in C \times E$, it remains to establish that $|J_{11,i}| = 2^{-i}p_i(z)/3$ for every $i \in \mathbb{N}$. To do so, we analyze the last constraint in Figure 21.

Almost every choice of the C-root x and the D_G-root x' in the last constraint in Figure 21 satisfies that $\eta_C(x) \in [0, 1/3)$, $\eta_{D_G}(x') \in [2/3, 1)$ and $[\![3\eta_C(x)]\!] = [\![3\eta_{D_G}(x')]\!]$. Fix such a choice of roots and let $i = [\![3\eta_C(x)]\!] = [\![3\eta_{D_G}(x')]\!]$. Recalling that $C \times \eta_{D_G}^{-1}([2/3, 1))$ encodes the coefficients $\pi^+_{i,j}, \pi^-_{i,j}$ of the polynomials, we can see that the constraint implies that $|J_{11,i}|$ is equal to

$$\sum_{k \in \mathbb{N}} 9 \cdot 2^{2k} \pi^+_{i,k} \cdot \frac{1}{3 \cdot 2^k} \cdot \frac{z^{M_k}}{3 \cdot 2^k} \cdot \frac{1}{3 \cdot 2^i} - \sum_{k \in \mathbb{N}} 9 \cdot 2^{2k} \pi^-_{i,k} \cdot \frac{1}{3 \cdot 2^k} \cdot \frac{z^{M_k}}{3 \cdot 2^k} \cdot \frac{1}{3 \cdot 2^i} = \frac{p_i(z)}{3 \cdot 2^i}\,.$$

It follows that $|J_{11,i}| = \frac{p_i(z)}{3 \cdot 2^i}$ for every $i \in \mathbb{N}$. Hence, we can conclude that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in C \times E$. Since we have already established that $3 \cdot 2^i \cdot |J_{11,i}| \in [l_i, u_i]$, we also obtain that $p_i(z) \in [l_i, u_i]$.

6.7 Proof of Theorem 21 – cleaning up

In Subsections 6.1–6.6, we showed that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in (A \cup B \cup C \cup D_A \cup \ldots \cup D_G \cup E \cup F)^2$. We now focus on the remaining tiles and consider the decorated constraints depicted in Figures 22 and 23.

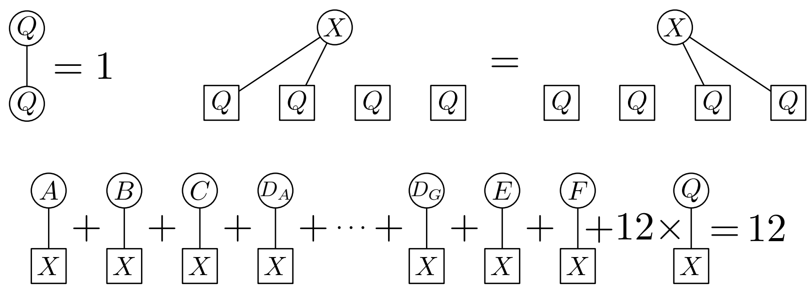

Figure 22: Decorated constraints forcing the structure of the tiles $X \times Q$ and $Q \times Q$, where $X \in \{A, B, C, D_A, \ldots, D_G, E, F\}$.

The first constraint in Figure 22 implies that $W(x,y) = 1$ for almost every $(x,y) \in Q \times Q$. Next fix $X \in \{A, B, C, D_A, \ldots, D_G, E, F\}$ and consider the second constraint on the first line in Figure 22. The constraint yields that the following holds for almost all $x, x', x'', x''' \in Q$:

$$\int_X W(x,y)W(x',y)\,\mathrm{d}y = \int_X W(x'',y)W(x''',y)\,\mathrm{d}y\,,$$

which implies by Lemma 7 that there exists $K \in \mathbb{R}$ such that

$$\int_X W(x,y)W(x',y)\,\mathrm{d}y = K \qquad \text{and} \qquad \int_X W(x,y)^2\,\mathrm{d}y = K$$

for almost all $x, x', \in Q$. However, this is only possible if there exists a measurable function $\xi : X \to [0,1]$ such that $W(x,y) = \xi(y)$ for almost all $(x,y) \in Q \times X$. In particular, $\deg^Q(y) = \xi(y)$ for almost all $y \in X$. The constraint on the second line implies that

$$\xi(y) = \frac{12 - \deg^A(y) - \deg^B(y) - \ldots - \deg^E(y) - \deg^F(y)}{12}$$

for almost all $y \in X$. Since $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in (A \cup B \cup C \cup D_A \cup \ldots \cup D_G \cup E \cup F)^2$, we get that $\xi(y) = W_P(x, g(y))$ for every $x \in Q$ and almost every $y \in X$. We conclude that $W(x,y) = W_P(g(x), g(y))$ for almost all $(x,y) \in X \times Q$, where we recall that $X \in \{A, B, C, D_A, \ldots, D_G, E, F\}$.

Fix (X,k) to be one of the following pairs: $(A,1)$, $(B,2)$, $(C,3)$, $(D_A,4)$, $\ldots$, $(D_G,10)$, $(E,11)$, $(F,12)$, $(Q,25)$ and $(R,0)$. Consider the decorated constraints depicted in Figure 23. The two constraints imply that

$$\frac{1}{|R|}\int_R W(x,y)\,\mathrm{d}y = \frac{k}{25} \qquad \text{and} \qquad \frac{1}{|R|}\int_R W(x,y)W(x',y)\,\mathrm{d}y = \frac{k^2}{625}$$

for almost all $x, x' \in X$. The second equality yields by Lemma 7 that

$$\frac{1}{|R|}\int_R W(x,y)^2\,\mathrm{d}y = \frac{k^2}{625}$$

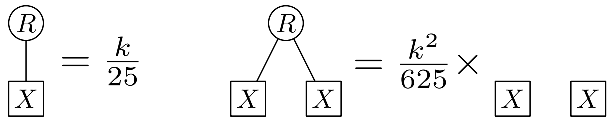

Figure 23: Decorated constraints forcing the structure of the tiles $X \times R$, where (X, k) is one of $(A, 1)$, $(B, 2)$, $(C, 3)$, $(D_A, 4)$, $\dots$, $(D_G, 10)$, $(E, 11)$, $(F, 12)$, $(Q, 25)$ and $(R, 0)$.

for almost all $x \in X$. Hence, the Cauchy-Schwarz Inequality implies that it holds for almost all $x \in X$ that $W(x, y)$ as a function of $y \in R$ is constant almost everywhere on R. It follows that $W(x, y) = k/25$ for almost all $(x, y) \in X \times R$. We can now conclude that $W(x, y) = W_P(g(x), g(y))$ for almost all $(x, y) \in [0, 1) \times [0, 1)$, which finishes the proof of Theorem 21.

7 Dependence on parameters

In this section, we analyze the dependence of the structure of the graphon $W_P(z)$ and the densities of graphs in this graphon on a bounding sequence P and a vector $z \in [0, 1]^{\mathbb{N}}$. We start with an additional definition: the L_1-distance between two bounding sequences P and P' is equal to

$$\sup_{z \in [0,1]^{\mathbb{N}}} \int_{[0,1)^2} |W_P(z)(x, y) - W_{P'}(z)(x, y)| \, \mathrm{d}x \, \mathrm{d}y.$$

We remark that we view the graphons in the definition above purely as functions from $[0, 1)^2$ to $\mathbb{R}$, i.e., we do not consider rearrangements of their underlying spaces as in the standard definition of the L_1-distance between graphons.

Lemma 22. *Let P be a bounding sequence. For every $\varepsilon > 0$, there exists an integer k_P such that the L_1-distance between P and any k_P-strengthening of P is at most ε.*

Proof. Fix $z \in [0, 1]^{\mathbb{N}}$ and a strengthening P' of the bounding sequence P. We will again just write W_P and $W_{P'}$ instead of $W_P(z)$ and $W_{P'}(z)$, respectively. Our goal is to show that if P' is a k_P-strengthening of P, for a sufficiently large integer k_P, which does not depend on P' and z, then the L_1-distance between W_P and $W_{P'}$, viewed as functions from $[0, 1)^2$ to $\mathbb{R}$, is at most ε. Note that the graphons W_P and $W_{P'}$ can differ only in the following tiles: $(D_A \cup \ldots \cup D_G)^2$, $C \times E$, $C \times D_G$, $D_G \times E$ and $X \times Q$, where $X \in \{C, D_A, \dots, D_G, E\}$.

Let k_0 be the integer from Proposition 3 applied for $\varepsilon/6$. Furthermore, let W_F be the graphon for the bounding sequence P from the definition of W_P, and let W'_F be the graphon for the bounding sequence P'. For a dyadic square S, let $d_{W_F}(S)$ be its density in W_F, i.e.,

$$d_{W_F}(S) = \frac{\int_S W_F(x,y)\,\mathrm{d}x\,\mathrm{d}y}{|S|}.$$

Finally, let ε_F be equal to the minimum of the values

$$\frac{\lfloor 2^{k_0} d_{W_F}(S) \rfloor + 1}{2^{k_0}} - d_{W_F}(S),$$

taken over all dyadic squares S of sizes $2^0, 2^{-1}, \ldots, 2^{-k_0}$. Note that ε_F is positive. Set k_1 to be an integer such that $2^{-k_1} < 2^{-k_0} \cdot \varepsilon_F$. Note that the choice of k_1 depends only on the bounding sequence P and the value of ε. If the bounding sequences P and P' agree on the first k_1 elements, then the definition of the graphons W_F and $W_{F'}$ imply that their L_1-distance is at most 2^{-k_1}. In addition, if P' is a strengthening of P, then $W'_F(x,y) \geq W_F(x,y)$ for all $(x,y) \in [0,1)^2$. Hence, if P' is a k_1-strengthening of P, then the density of each dyadic square of size at least 2^{-k_0} in W'_F is at least the density of that square in W_F and the difference between the two densities is at most ε_F. In particular, the densities of the dyadic squares of $2^0, 2^{-1}, \ldots, 2^{-k_0}$ in W_F and W'_F agree up to the first k_0 bits after the decimal point in the standard binary representation. Consequently, the L_1-distance between the graphons W_0 and W'_0 from Theorem 2 applied with W_F and W'_F, respectively, is at most $\varepsilon/6$. We conclude that if P' is a k_1-strengthening of P, then

$$\int_{(D_A \cup \ldots \cup D_G)^2} |W_P(x,y) - W_{P'}(x,y)|\,\mathrm{d}x\,\mathrm{d}y \leq \frac{\varepsilon}{6}\,.$$

Next, set k_2 to be an integer such that $2^{-k_2} \leq \varepsilon/6$. Observe that if the bounding sequences P and P' agree on the first k_2 elements, then the measure of the points in each of the tiles $C \times E$, $C \times D_G$ and $D_G \times E$ where W_P and $W_{P'}$ differ is at most 2^{-k_2} of the total measure of the tile. We set $k_P = \max\{k_1, k_2\}$ and conclude that if P' is a k_P-strengthening of P, then

$$\int_{(A \cup B \cup C \cup D_A \cup \ldots \cup D_G \cup E \cup F)^2} |W_P(x,y) - W_{P'}(x,y)|\,\mathrm{d}x\,\mathrm{d}y \leq \frac{\varepsilon}{3}\,. \tag{6}$$

Fix $x \in A \cup B \cup C \cup D_A \cup \ldots \cup D_G \cup E \cup F$. The definition of the graphons W_P and $W_{P'}$ yields that

$$\int_{A \cup B \cup C \cup D_A \cup \ldots \cup D_G \cup E \cup F \cup Q} W_P(x,y) - W_{P'}(x,y)\,\mathrm{d}y = 0.$$

Since the value of $W_P(x,y)$ is the same for all $y \in Q$ and the value of $W_{P'}(x,y)$ is also the same for all $y \in Q$, it follows that

$$\int_Q |W_P(x,y) - W_{P'}(x,y)|\,\mathrm{d}y = \left| \int_Q W_P(x,y) - W_{P'}(x,y)\,\mathrm{d}y \right|$$

$$= \left| \int_{A \cup B \cup C \cup D_A \cup \ldots \cup D_G \cup E \cup F} W_P(x,y) - W_{P'}(x,y) \, \mathrm{d}y \right|$$
$$\leq \int_{A \cup B \cup C \cup D_A \cup \ldots \cup D_G \cup E \cup F} |W_P(x,y) - W_{P'}(x,y)| \, \mathrm{d}y. \tag{7}$$

We derive from (6) and (7) that

$$\int_{(A \cup B \cup C \cup D_A \cup \ldots \cup D_G \cup E \cup F) \times Q} |W_P(x,y) - W_{P'}(x,y)| \, \mathrm{d}x \, \mathrm{d}y \leq \frac{\varepsilon}{3}. \tag{8}$$

The symmetric estimate holds for the integral over $Q \times (A \cup B \cup C \cup D_A \cup \ldots \cup D_G \cup E \cup F)$. Since the graphons W_P and $W_{P'}$ agree on the tile $Q \times Q$ and all tiles involving the part R, we conclude using the estimates (6) and (8) that their L_1-distance is at most ε. □

We next analyze the dependence of the density of a graph H in a graphon $W_P(z)$ on the vector z. We start by showing that this density is a countable sum of polynomials in z, where the polynomials appearing in the sum have their coefficients sufficiently restricted. In the statement of the next lemma, we use the linear order on multisets of natural numbers defined in Section 3, i.e., M_i is the i-th multiset in this order.

Lemma 23. *For every graph H, there exists a countable set $\mathcal{H}$, a constant $c_H \in (0,1)$ and constants $\beta_{H',i}$, $H' \in \mathcal{H}$ and $i \in \mathbb{N}$, with the following properties. For every bounding sequence $P = (p_j, l_j, u_j)_{j \in \mathbb{N}}$, there exist polynomials*

$$q_{P,H'}(z) = \sum_{i \in \mathbb{N}} \alpha_{P,H',i} \cdot z^{M_i} \tag{9}$$

such that

$$\tau(H, W_P(z)) = \sum_{H' \in \mathcal{H}} q_{P,H'}(z) \tag{10}$$

for every $z \in [0,1]^{\mathbb{N}}$ that satisfies $p_j(z) \in [l_j, u_j]$ for all $j \in \mathbb{N}$. In addition, it holds that $|\alpha_{P,H',i}| \leq \beta_{H',i}$ for every $H' \in \mathcal{H}$ and $i \in \mathbb{N}$, the sum $\sum_{H' \in \mathcal{H}} \beta_{H',i}$ is finite for all $i \in \mathbb{N}$, and is at most $c_H^{\Sigma(M_i)}$ for all but finitely many $i \in \mathbb{N}$.

Furthermore, for every $H' \in \mathcal{H}$ and every real $\varepsilon > 0$, there exist an integer $k_{P,H'}$ and a real $\varepsilon_{P,H'} > 0$ such that if P' is a bounding sequence that agrees with P on the first $k_{P,H'}$ elements and that has L_1-distance at most $\varepsilon_{P,H'}$ from P, then the polynomials $q_{P,H}(z)$ and $q_{P',H}(z)$ are ε-close.

Proof. Fix a graph H and a bounding sequence P for the proof of the lemma; let $v_1, \ldots, v_n$ be the vertices of H. We start by defining the set $\mathcal{H}$. Let $C_{3i+j} \subseteq C$ be defined as follows:

$$C_{3i+j} = \iota_C \left(\left[\frac{j - 2^{-i}}{3}, \frac{j - 2^{-(i+1)}}{3} \right) \right),$$

where $(i,j) \in \mathbb{N}_0 \times \{1,2,3\}$. Note that the graphon $W_P(z)$ is constant on every set $C_k \times C_{k'}$, $k,k' \in \mathbb{N}$, and it is zero on the tile $C \times C$ outside the sets $C_{3k+1} \times C_{3k+1}$, $C_{3k+1} \times C_{3k+2}$, $C_{3k+2} \times C_{3k+1}$ and $C_{3k+2} \times C_{3k+2}$, $k \in \mathbb{N}$. We also define $E_{3i+j} \subseteq E$ as

$$E_{3i+j} = \iota_E\left(\left[\frac{j-2^{-i}}{3}, \frac{j-2^{-(i+1)}}{3}\right)\right)$$

for $(i,j) \in \mathbb{N}_0 \times \{1,2,3\}$ and

$$E_{3i+1,j} = \begin{cases} \iota_E\left(\left[\frac{1-2^{-i}}{3}, \frac{1-2^{-i}+l_i 2^{-(i+1)}}{3}\right)\right) & \text{if } j=1, \\ \iota_E\left(\left[\frac{1-2^{-i}+l_i 2^{-(i+1)}}{3}, \frac{1-2^{-i}+u_i 2^{-(i+1)}}{3}\right)\right) & \text{if } j=2, \\ \iota_E\left(\left[\frac{1-2^{-i}+u_i 2^{-(i+1)}}{3}, \frac{1-2^{-(i+1)}}{3}\right)\right) & \text{if } j=3, \end{cases}$$

for $(i,j) \in \mathbb{N} \times \{1,2,3\}$. Finally, let

$$D_{G,j} = \iota_{D_G}\left([(j-1)/3, j/3)\right)$$

for $j \in \{1,2,3\}$. We define the set $\mathcal{H}$ to be the set containing all copies of H where each vertex is labeled with one of the following sets: A, B, C_k, $D_A, \dots, D_F$, $D_{G,1}$, $D_{G,2}$, $D_{G,3}$, E_k, F, Q and R, $k \in \mathbb{N}$.

Fix $H' \in \mathcal{H}$ and let X_i be the label of the vertex v_i of H'. Form another graph G on the vertices of H as follows. For each vertex v_i, sample x_i from X_i uniformly and independently, and join the vertices v_i and $v_{i'}$ with probability equal to the value of the graphon $W_P(z)$ for $(x_i, x_{i'})$. We next show that the probability that the graph G is the graph H (preserving vertices) is a polynomial $p_{P,H'}(z)$ of z.

Let V_C and V_E consist of the vertices of H' that are labeled with C_k or E_k for some $k \in \mathbb{N}$, respectively, and let $V_{CE} = V_C \cup V_E$. Given two vertices of H' such that at least one of them is not contained in V_{CE}, the probability that they are joined by an edge does not depend on $z \in [0,1]^{\mathbb{N}}$ by Lemma 19. Let $s_{P,H'}$ be the probability that all such pairs of vertices have the same adjacencies in G and H. Furthermore, let $r_{P,H'}(z)$ be the probability that G and H agree on V_{CE} conditioned on the event that the graphs G and H agree outside V_{CE}. Clearly $p_{P,H'}(z) = s_{P,H'} \cdot r_{P,H'}(z)$. Since for all $y \in [0,1) \setminus (C \cup E)$, the value of $W_P(z)$ for (x,y) and (x',y) is the same if x and x' belong to the same set C_k, $E_{3k+1,1}$, $E_{3k+1,2}$, $E_{3k+1,3}$, E_{3k+2} or E_{3k+3}, $k \in \mathbb{N}$, the event that G and H agree outside V_{CE} does not restrict the values x_i associated with vertices labeled with C_k, E_{3k+2} or E_{3k+3}, however, it may restrict the values x_i associated with vertices labeled with E_{3k+1} to be (uniformly chosen) from $E_{3k+1,1}$, $E_{3k+1,2}$, $E_{3k+1,3}$ or from a union of some of these three sets.

We next show that the function $r_{P,H'}(z)$ is a polynomial in z. Two vertices of H that are labeled with C_k and $C_{k'}$, $k,k' \in \mathbb{N}$, are joined by an edge with probability equal to 0, z^M or $1-z^M$ for a finite multiset M of positive integers; the event that such a pair of vertices is joined by an edge is independent of the rest of the structure of G. The probability that a vertex labeled with E_{3k+1}, E_{3k+2} or E_{3k+3}, $k \in \mathbb{N}$, has the same adjacencies in G and H is equal to one of the following expressions: 0, 1, $p_j(z)$, $1-p_j(z)$, $p_j(z)/u_j$, $(u_j-p_j(z))/u_j)$, $(p_j(z)-l_j)/(1-l_j)$, $(1-p_j(z))/(1-l_j)$, $(p_j(z)-l_j)/(u_j-l_j)$ and $(u_j-p_j(z))/(u_j-l_j)$, $j \in \mathbb{N}$; here, we use that $p_j(z) \in [l_j, u_j]$ for every $j \in \mathbb{N}$. Moreover, these events are independent of each other because the graphon $W_P(z)$ is zero on the tile $E \times E$, and they are also independent of the adjacencies between the vertices labeled with C_k, $k \in \mathbb{N}$, because for every $y \in E$, the value of $W_P(z)$ for (x, y) and (x', y) is the same if x and x' are from the same C_k, $k \in \mathbb{N}$. Hence, $r_{P,H'}(z)$ is a product of the following expressions: 0, 1, z^M, $1-z^M$, $p_j(z)$, $1-p_j(z)$, $p_j(z)/u_j$, $(u_j-p_j(z))/u_j)$, $(p_j(z)-l_j)/(1-l_j)$, $(1-p_j(z))/(1-l_j)$, $(p_j(z)-l_j)/(u_j-l_j)$ and $(u_j-p_j(z))/(u_j-l_j)$, where M is a finite multiset of positive integers and $j \in \mathbb{N}$. Moreover, for each expression that appears with a denominator of $1-l_j$, u_j or u_j-l_j, there is a corresponding term in the product defining the constant $s_{P,H'}$. We conclude that $p_{P,H'}(z)$ is a product of a constant from $[0, 1]$ and terms equal to one of the following expressions: 0, 1, z^M, $1-z^M$, $p_j(z)$, $1-p_j(z)$, $u_j-p_j(z)$ and $p_j(z)-l_j$, where M is a finite multiset of positive integers and $j \in \mathbb{N}$. Set

$$q_{P,H'}(z) = p_{P,H'}(z) \cdot \prod_{i=1}^{n} |X_i| \,. \tag{11}$$

The definition of $\tau(H, W_P)$ now implies that

$$\tau(H, W_P(z)) = \sum_{H' \in \mathcal{H}} q_{P,H'}(z)$$

for every $z \in [0, 1]^{\mathbb{N}}$ that satisfies $p_j(z) \in [l_j, u_j]$ for all $j \in \mathbb{N}$.

We next establish the existence of the constants $\beta_{H',i}$. Fix a multiset M_i of positive integers. We are going to trace how the monomial z^{M_i} can appear in the product defining the polynomial $p_{P,H'}(z)$ for $H' \in \mathcal{H}$.

Recall that V_C and V_E are the sets vertices of H' labeled with C_k, $k \in \mathbb{N}$ and E_k, $k \in \mathbb{N}$, respectively. In the analysis above, we expressed $p_{P,H'}(z)$ as a product of terms corresponding to vertices of V_E and terms corresponding to pairs of vertices from V_C. So, consider a partition $\mathcal{P}$ of M_i into $|V_E| + \binom{|V_C|}{2} \le n^2$ multisets M^{v_j}, $v_j \in V_E$, and $M^{v_j v_{j'}}$, $\{v_j, v_{j'}\} \subseteq V_C$. Notice that for $k > 0$, a pair of vertices v_j and $v_{j'}$ from V_C can contribute to the product defining $p_{P,H'}(z)$ by the monomial z^{M_k} only if both v_j and $v_{j'}$ are labeled by C_{3k+1} or C_{3k+2}. So, we say that a partition $\mathcal{P}$ is *consistent* with H' if for each pair $v_j, v_j' \in V_C$ either $M^{v_j v_j'}$ is the empty set or $v_j, v_{j'} \in C_{3k+1} \cup C_{3k+2}$ and $M^{v_j v_j'} = M_k$. Let $\mathscr{P}_{H',i}$ be the set of all partitions of M_i consistent with H'.

Observe that if a vertex $v_j \in V_E$ contributes to the product defining $p_{P,H'}(z)$ by the monomial z^{M_k}, then the corresponding coefficient in the product is at most

2^{-2^k}. In particular, it is at most $2^{-2^{\Sigma(M_k)}}$ by Observation 8. We next set

$$\beta_{H',i} = \prod_{j=1}^{n} |X_j| \sum_{\mathcal{P} \in \mathscr{P}_{H',i}} \prod_{v_{j'} \in V_E} 2^{-2^{\Sigma(M^{v_{j'}})}}.$$

The analysis above implies that $|\alpha_{P,H',i}| \leq \beta_{H',i}$.

We next prove the existence of the constant c_H and that the sum $\sum_{H' \in \mathcal{H}} \beta_{H',i}$ is finite for all $i \in \mathbb{N}$. Consider a partition $\mathcal{P} \in \mathscr{P}_{H',i}$ and let

$$S = \sum_{v_j \in V_E} \Sigma(M^{v_j}).$$

Note that there must exist a pair of vertices $v_j, v_{j'} \in V_C$ with $M^{v_j v_{j'}} = M_k$ such that $k \geq \Sigma(M_k) \geq (\Sigma(M_i) - S)/n^2$ (the first inequality holds by Observation 8). So, every partition of M_i consistent with $H' \in \mathcal{H}$ can be constructed as follows: we first choose an integer S between 0 and $\Sigma(M_i)$. We then choose which pairs of vertices from V_C correspond to the empty set and which to the unique multiset M_k such that both vertices in the pair are labeled by C_{3k+1} or C_{3k+2}. This can be done in at most 2^{n^2} ways. Then, the remaining at most S elements of M_i are distributed among the sets M^{v_j}, $v_j \in V_E$, in at most n^S ways.

We are now ready to estimate the sum of $\beta_{H',i}$. This will be done by considering each $H' \in \mathcal{H}$ together with all partitions of M_i that are consistent with H'. First, we choose which vertices of H will belong to V_C and to V_E; this can be done in 3^n ways. Next, if V_C is non-empty, we choose i_C such that the vertex $v_{i_C} \in V_C$ is labeled with C_{3k+1} or C_{3k+2} for the largest value of k among the vertices from V_C; the index i_C can be chosen in at most n ways. Then, we choose the value of S and consider each $H' \in \mathcal{H}$ such that the vertex v_{i_C} is labeled with C_{3k+1} or C_{3k+2} for some $k \geq (\Sigma(M_i) - S)/n^2$. Finally, we choose a partition of M_i that is consistent with H'. We established above that this can be chosen in at most $2^{n^2} \cdot n^S$ ways. Observe that if V_E is non-empty, then there must be a vertex $v_j \in V_E$ such that $\Sigma(M^{v_j}) \geq S/n$, in particular, the last product in the definition of $\beta_{H',i}$ is at most $2^{-2^{S/n}}$. If V_E is empty, which implies $S = 0$, the bound $2^{-2^{S/n}}$ is off by a factor of two, however, this is compensated for by an overestimate in the term 2^{n^2}, so the bound $2^{n^2} n^S 2^{-2^{S/n}}$ is still valid. We conclude that

$$\begin{aligned}
\sum_{H' \in \mathcal{H}} \beta_{H',i} \leq & \sum_{H' \in \mathcal{H}} \prod_{j=1}^{n} |X_j| \sum_{S=0}^{\Sigma(M_i)} 2^{n^2} n^S 2^{-2^{S/n}} \\
\leq & n \cdot 3^n \sum_{S=0}^{\Sigma(M_i)} \sum_{k \geq \frac{\Sigma(M_i) - S}{n^2}} \sum_{\substack{H' \in \mathcal{H} \\ X_{i_C} \in \{C_{3k+1}, C_{3k+2}\}}} 2^{n^2} \cdot n^S \cdot 2^{-2^{S/n}} \prod_{j=1}^{n} |X_j| \\
= & n \cdot 3^n \cdot 2^{n^2} \sum_{S=0}^{\Sigma(M_i)} n^S \cdot 2^{-2^{S/n}} \sum_{k \geq \frac{\Sigma(M_i) - S}{n^2}} \sum_{\substack{H' \in \mathcal{H} \\ X_{i_C} \in \{C_{3k+1}, C_{3k+2}\}}} \prod_{j=1}^{n} |X_j|
\end{aligned}$$

$$
\begin{aligned}
=& n \cdot 3^n \cdot 2^{n^2} \sum_{S=0}^{\Sigma(M_i)} n^S \cdot 2^{-2^{S/n}} \sum_{k \geq \frac{\Sigma(M_i)-S}{n^2}} 2|C_{3k+1}| \sum_{\substack{H' \in \mathcal{H} \\ X_{i_C} \in \{C_{3k+1}, C_{3k+2}\}}} \prod_{j \neq i_C} |X_j| \\
\leq& n \cdot 3^n \cdot 2^{n^2} \sum_{S=0}^{\Sigma(M_i)} n^S \cdot 2^{-2^{S/n}} \cdot 2^{-(\Sigma(M_i)-S)/n^2} \\
\leq& n \cdot 3^n \cdot 2^{n^2} \sum_{S=0}^{\Sigma(M_i)} 2^{nS - 2^{S/n} - \Sigma(M_i)/n^2}. \qquad (12)
\end{aligned}
$$

In particular, the sum is finite for every $i \in \mathbb{N}$.

We next bound the exponents in the summands in (12). If $S \leq \Sigma(M_i)/(2n^3)$, then we have:

$$
nS - 2^{S/n} - \frac{\Sigma(M_i)}{n^2} \leq nS - \frac{\Sigma(M_i)}{n^2} \leq -\frac{\Sigma(M_i)}{2n^2}.
$$

If $S \geq \Sigma(M_i)/(2n^3)$ and $2nS \leq 2^{S/n}$, then we obtain the following:

$$
nS - 2^{S/n} - \frac{\Sigma(M_i)}{n^2} \leq nS - 2^{S/n} \leq -nS \leq -\frac{\Sigma(M_i)}{2n^2}.
$$

Since n is the number of vertices of H, i.e., n is fixed, there exists Σ_0 such that if $\Sigma(M_i) \geq \Sigma_0$, then any $S \geq \Sigma(M_i)/(2n^3) \geq \Sigma_0/(2n^3)$ satisfies $2nS \leq 2^{S/n}$. We conclude that there exists $c_H \in (0,1)$ such that

$$
n \cdot 3^n \cdot 2^{n^2} \cdot \sum_{S=0}^{\Sigma(M_i)} 2^{nS - 2^{S/n} - \Sigma(M_i)/n^2} \leq n \cdot 3^n \cdot 2^{n^2} \cdot (\Sigma(M_i)+1) \cdot 2^{-\frac{\Sigma(M_i)}{2n^2}} \leq c_H^{\Sigma(M_i)}
$$

if $\Sigma(M_i)$ is sufficiently large.

It now remains to establish the last part of the statement of the lemma. Fix $H' \in \mathcal{H}$ and let $\varepsilon > 0$ be given. Let $k_{P,H'}$ be the largest index k such that a vertex of H' is labeled with E_{3k+1} or E_{3k+2}. If two bounding sequences P and P' agree on the first $k_{P,H'}$ elements, then the polynomials $r_{P,H'}(z)$ and $r_{P',H'}(z)$ are the same. Let D be such that the values of $r_{P,H'}(z)$ and all its first partial derivatives on $[0,1]^{\mathbb{N}}$ belong to $[-D, D]$. Next, choose $\varepsilon_{P,H'}$ small enough such that if the L_1-distance of the graphons $W_P(z)$ and $W_{P'}(z)$ is at most $\varepsilon_{P,H'}$, then the values $s_{P,H'}$ and $s_{P',H'}$ differ by at most ε/D (regardless of the choice of $z \in [0,1]^{\mathbb{N}}$). For this choice of $k_{P,H'}$ and $\varepsilon_{P,H'}$, the polynomials $q_{P,H'}(z)$ and $q_{P',H'}(z)$ are ε-close if the L_1-distance of P and P' is at most $\varepsilon_{P,H'}$ and P and P' agree on the first $k_{P,H'}$ elements. This finishes the proof of the lemma. □

We emphasize that $\tau(H, W_P(z))$ and the right side of (10) are not necessarily equal for all $z \in [0,1]^{\mathbb{N}}$, in particular, for those $z \in [0,1]^{\mathbb{N}}$ that do not satisfy the constraints implied by the bounding sequence P.

We next show that the density of each graph H in $W_P(z)$ is equal to a totally analytic function of z for those $z \in [0,1]^{\mathbb{N}}$ that satisfy the constraints implied by the bounding sequence P. We start with the following simple observation.

Lemma 24. *Let M_i be the i-th multiset in the linear order defined in Section 3. For every real $c \in (0,1)$ and integer $k \in \mathbb{N}_0$, the sum*

$$\sum_{i=1}^{\infty} |M_i|^k c^{\Sigma(M_i)}$$

is finite.

Proof. We group the summands into groups with the same value of $\Sigma(M_i)$. The number of multisets M_i with $\Sigma(M_i) = n$ is the number of ways that an integer n can be expressed as a sum of positive integers (where the order does not matter). This number is well-known to be at most $e^{10\sqrt{n}}$ if $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Since $|M_i| \leq \Sigma(M_i)$, the sum from the statement of the lemma is at most

$$\sum_{n=1}^{n_0-1} \sum_{M_i, \Sigma(M_i)=n} |M_i|^k c^{\Sigma(M_i)} + \sum_{n=n_0}^{\infty} e^{10\sqrt{n}} n^k c^n. \tag{13}$$

Since the first sum in (13) consists of finitely many summands, the sum from the statement of the lemma is finite. □

We also need the following theorem, which can be found, for example, in [49, Theorem 7.17].

Theorem 25. *Let $I = [a,b]$ and $f_n : I \to \mathbb{R}$ be a sequence of functions differentiable on I. If f'_n converges uniformly to a function g on I and there exists a point x_0 in the interior of I such that $f_n(x_0)$ converges, then the functions f_n converge uniformly to a differentiable function $f : I \to \mathbb{R}$ and $f' = g$.*

We are now ready to state a lemma that will guarantee that the functions $t_{P,H}(z)$ defined further are totally analytic.

Lemma 26. *For $i \in \mathbb{N}$, let M_i be the i-th multiset in the linear order defined in Section 3, and let α_i be a real number. Suppose that there exists $c \in (0,1)$ such that $|\alpha_i| \leq c^{\Sigma(M_i)}$ for all but finitely many $i \in \mathbb{N}$. Then the function $t : [0,1]^{\mathbb{N}} \to \mathbb{R}$ defined as*

$$t(z) = \sum_{i=1}^{\infty} \alpha_i z^{M_i} \tag{14}$$

is well-defined on $[0,1]^{\mathbb{N}}$ and totally analytic. Moreover, for any multiset M, the partial derivatives can be expressed as

$$\frac{\partial^k}{(\partial z)^M} t(z) = \sum_{i=1}^{\infty} \alpha_i \frac{\partial^k}{(\partial z)^M} z^{M_i}. \tag{15}$$

Finally, the infinite sums in (14) and (15) converge uniformly on $[0,1]^{\mathbb{N}}$.

Proof. Fix $\varepsilon > 0$ such that $c(1+\varepsilon) < 1-\varepsilon$. Observe that if $z \in [-\varepsilon, 1+\varepsilon]^{\mathbb{N}}$, then $|z^{M_i}| \le (1+\varepsilon)^{|M_i|} \le (1+\varepsilon)^{\Sigma(M_i)}$ for every i, in particular, $|\alpha_i z^{M_i}| \le (1-\varepsilon)^{\Sigma(M_i)}$ for all but finitely many $i \in \mathbb{N}$. Lemma 24 applied with $1-\varepsilon$ and $k=0$ yields that the infinite sum defining the function t converges uniformly on $[-\varepsilon, 1+\varepsilon]^{\mathbb{N}}$, in particular, the function t is well-defined on $[0,1]^{\mathbb{N}}$ and the sum (14) converges uniformly on $[0,1]^{\mathbb{N}}$.

Fix a finite multiset M of positive integers and define β_i to be the coefficient of the derivative of $\alpha_i z^{M_i}$ with respect to all z_j, $j \in M$. Observe that $\beta_i = 0$ whenever M is not a subset of M_i. We claim that

$$\frac{\partial^k}{(\partial z)^M} t(z) = \sum_{i=1}^{\infty} \beta_i z^{M_i \setminus M}, \tag{16}$$

where $k = |M|$. Since $|\beta_i| \le |\alpha_i| \cdot |M_i|^k$ and $|z^{M_i \setminus M}| \le (1+\varepsilon)^{\Sigma(M_i)}$ for every $z \in [-\varepsilon, 1+\varepsilon]^{\mathbb{N}}$, we get that $|\beta_i z^{M_i \setminus M}| \le |M_i|^k \cdot (1-\varepsilon)^{\Sigma(M_i)}$ for every such z and for all but finitely many $i \in \mathbb{N}$. Hence, Lemma 24 applied with $1-\varepsilon$ and k implies that the right side of (16) uniformly converges on $[-\varepsilon, 1+\varepsilon]^{\mathbb{N}}$. Let $g(z)$ be its limit for $z \in [-\varepsilon, 1+\varepsilon]^{\mathbb{N}}$.

By induction on k, we show that (16) holds for all multisets M. Since the base of the induction and the induction step follow the same line of arguments, we just focus on the case $k=1$, i.e., $M = \{j\}$ for some $j \in \mathbb{N}$. Let $f_n(z)$ be the sum of the first n terms on the right side in (14). Since $\frac{\partial}{\partial z_j} f_n(z)$ is the sum of the first n terms on the right side in (16), $\frac{\partial}{\partial z_j} f_n(z)$ converges to $g(z)$. Since f_n converges to f everywhere on $[-\varepsilon, 1+\varepsilon]^{\mathbb{N}}$, we conclude using Theorem 25 (applied to each fixed choice of $z_i \in [-\varepsilon, 1+\varepsilon]$, $i \notin M$) that $\frac{\partial}{\partial z_j} t(z)$ is $g(z)$. It follows that the function $t(z)$ is totally analytic. $\square$

We are now ready to define the function $t_{P,H}(z)$. Fix a bounding sequence P and a graph H, and let M_i be the i-th multiset in the linear order defined in Section 3. We apply Lemma 23 and let $\alpha_{P,i}$ be the sum of the coefficients $\alpha_{P,H',i}$ of z^{M_i} in the polynomials $q_{P,H'}(z)$ in (10), where the sum is taken over $H' \in \mathcal{H}$. Note that there exists $c_H \in (0,1)$ such that $|\alpha_{P,i}| \le c_H^{\Sigma(M_i)}$ for all but finitely many $i \in \mathbb{N}$. By Lemma 26, the function

$$t_{P,H}(z) = \sum_{i=1}^{\infty} \alpha_{P,i} z^{M_i}$$

is well-defined on $[0,1]^{\mathbb{N}}$ and totally analytic.

We are now ready to prove Lemma 10.

Proof of Lemma 10. We have already shown that the function $t_{P,H}(z)$ is totally analytic. Hence, we need to show that $t_{P,H}(z) = \tau(H, W_P(z))$ for every $z \in [0,1]^{\mathbb{N}}$

that satisfies $p_j(z) \in [l_j, u_j]$ for all $j \in \mathbb{N}$. By Lemma 23, we need to show that

$$t_{P,H}(z) = \sum_{H' \in \mathcal{H}} q_{P,H'}(z),$$

where $\mathcal{H}$ is the countable set $\mathcal{H}$ from the statement the lemma. This is equivalent to establishing that

$$t_{P,H}(z) = \sum_{i=1}^{\infty} \alpha_{P,i} z^{M_i} = \sum_{i=1}^{\infty} \sum_{H' \in \mathcal{H}} \alpha_{P,H',i} z^{M_i} = \sum_{H' \in \mathcal{H}} \sum_{i=1}^{\infty} \alpha_{P,H',i} z^{M_i} = \sum_{H' \in \mathcal{H}} q_{P,H'}(z),$$

where $\alpha_{P,H',i}$ is the coefficient of z^{M_i} in $q_{P,H'}(z)$. The first, second and fourth equalities follow from the definitions. The third equality holds because the sum of the absolute values of the terms is finite. Indeed, since $z \in [0,1]^{\mathbb{N}}$, we have

$$\sum_{i=1}^{\infty} \sum_{H' \in \mathcal{H}} \left| \alpha_{P,H',i} z^{M_i} \right| \le \sum_{i=1}^{\infty} \sum_{H' \in \mathcal{H}} \left| \alpha_{P,H',i} \right|. \tag{17}$$

Since all but finitely many terms $\sum_{H' \in \mathcal{H}} |\alpha_{P,H',i}|$ are at most $c_H^{\Sigma(M_i)}$, which is at most c_H^i by Observation 8, the sum (17) is indeed finite. This completes the proof of the lemma. □

To prove Lemma 11, we need an additional lemma.

Lemma 27. *For every bounding sequence P, graph H, and real $\varepsilon > 0$, there exist a real $\varepsilon_{P,H} > 0$ and an integer $k_{P,H}$ with the following property. If P' is a bounding sequence that agrees with P on the first $k_{P,H}$ elements and has L_1-distance at most $\varepsilon_{P,H}$ from P, then the functions $t_{P,H}(z)$ and $t_{P',H}(z)$ are ε-close.*

Proof. We first show that there exists a finite subset $\mathcal{H}'$ of $\mathcal{H}$ that can be chosen independently of P such that the function $\widetilde{t}_{P,H}(z)$ defined as

$$\widetilde{t}_{P,H}(z) = \sum_{H' \in \mathcal{H}'} q_{P,H'}(z) \tag{18}$$

is $\varepsilon/4$-close to $t_{P,H}(z)$. Here $q_{P,H'}(z)$ are the polynomials (9) from the statement of Lemma 23. Recall that $|\alpha_{P,H',i}| \le \beta_{H',i}$, and for all but finitely $i \in \mathbb{N}$:

$$\sum_{H' \in \mathcal{H}} \beta_{H',i} \le c_H^{\Sigma(M_i)},$$

where $c_H \in (0,1)$ is the constant from the statement of Lemma 23. Let i_0 be such that

$$\sum_{i=i_0+1}^{\infty} \sum_{H' \in \mathcal{H}} i \cdot \beta_{H',i} \le \sum_{i=i_0+1}^{\infty} i \cdot c_H^{\Sigma(M_i)} \le \frac{\varepsilon}{16}.$$

Note that such an i_0 exists by Lemma 24 and can be chosen so that it does not depend on P. Finally choose $\mathcal{H}'$ to be a finite subset of $\mathcal{H}$ such that

$$\sum_{i=1}^{i_0} \sum_{H' \in \mathcal{H} \setminus \mathcal{H}'} \beta_{H',i} \cdot i \leq \frac{\varepsilon}{8}.$$

Note that such a set $\mathcal{H}'$ exists and can be chosen independently of P because for each $i \in \mathbb{N}$, the sum $\sum_{H' \in \mathcal{H}} \beta_{H',i}$ is finite. The choice of $\mathcal{H}'$ guarantees that for any bounding sequence P,

$$\sum_{i=1}^{i_0} \left| \alpha_{P,i} - \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} \right| \cdot i \sum_{i=1}^{i_0} \left| \sum_{H' \in \mathcal{H} \setminus \mathcal{H}'} \alpha_{P,H',i} \right| \cdot i \leq \frac{\varepsilon}{8},$$

where $\alpha_{P,i}$ is the coefficient of z^{M_i} in $t_{P,H}(z)$.

We now show that the functions $t_{P,H}(z)$ and $\widetilde{t}_{P,H}(z)$ are $\varepsilon/4$-close. Let $z \in [0,1]^{\mathbb{N}}$. Using (14), we have the following estimates:

$$\begin{aligned}
\left| t_{P,H}(z) - \widetilde{t}_{P,H}(z) \right| &= \left| \sum_{i=1}^{\infty} \alpha_{P,i} z^{M_i} - \sum_{i=1}^{\infty} \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} z^{M_i} \right| \\
&\leq \sum_{i=1}^{\infty} \left| \alpha_{P,i} - \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} \right| \cdot \left| z^{M_i} \right| \\
&\leq \sum_{i=1}^{\infty} \left| \alpha_{P,i} - \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} \right| \\
&\leq \frac{\varepsilon}{8} + \sum_{i=i_0+1}^{\infty} \left| \alpha_{P,i} - \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} \right| \\
&\leq \frac{\varepsilon}{8} + \sum_{i=i_0+1}^{\infty} |\alpha_{P,i}| + \sum_{i=i_0+1}^{\infty} \left| \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} \right| \\
&\leq \frac{\varepsilon}{8} + 2 \sum_{i=i_0+1}^{\infty} \sum_{H' \in \mathcal{H}} |\alpha_{P,H',i}| \leq \frac{\varepsilon}{4}.
\end{aligned}$$

Likewise, for any $k \in \mathbb{N}$, using (15), we obtain the following estimates for the derivative with respect to z_k:

$$\begin{aligned}
\left| \frac{\partial}{\partial z_k} t_{P,H}(z) - \frac{\partial}{\partial z_k} \widetilde{t}_{P,H}(z) \right| &= \left| \sum_{i=1}^{\infty} \alpha_{P,i} \frac{\partial}{\partial z_k} z^{M_i} - \sum_{i=1}^{\infty} \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} \frac{\partial}{\partial z_k} z^{M_i} \right| \\
&\leq \sum_{i=1}^{\infty} \left| \left(\alpha_{P,H,i} - \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} \right) \frac{\partial}{\partial z_k} z^{M_i} \right|
\end{aligned}$$

$$\begin{aligned}
&\le \sum_{i=1}^{\infty} \left| \alpha_{P,H,i} - \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} \right| \cdot |M_i| \\
&\le \frac{\varepsilon}{8} + \sum_{i=i_0+1}^{\infty} \left| \alpha_{P,H,i} - \sum_{H' \in \mathcal{H}'} \alpha_{P,H',i} \right| \cdot i \\
&\le \frac{\varepsilon}{8} + 2 \sum_{i=i_0+1}^{\infty} \sum_{H' \in \mathcal{H}'} i \cdot |\alpha_{P,H',i}| \le \frac{\varepsilon}{4} .
\end{aligned}$$

Let K be the size of $\mathcal{H}'$. For each $H' \in \mathcal{H}$, let $k_{P,H'}$ and $\varepsilon_{P,H'}$ be the values from the second part of the statement of Lemma 23 for $\varepsilon/2K$. Set $k_{P,H}$ to be the maximum of the values $k_{P,H'}$ and set $\varepsilon_{P,H}$ to be the minimum of the values $\varepsilon_{P,H'}$, in both cases taken over $H' \in \mathcal{H}'$. Note that the values of $k_{P,H}$ and $\varepsilon_{P,H}$ depend on P and H only.

Consider a bounding sequence P' that agrees with P on the first $k_{P,H}$ elements and that has L_1-distance from P at most $\varepsilon_{P,H}$. Let

$$\widetilde{t}_{P',H}(z) = \sum_{H' \in \mathcal{H}'} q_{P',H'}(z),$$

where $q_{P',H'}(z)$ are the polynomials from the statement of Lemma 23. Lemma 23 yields that the polynomials $q_{P,H'}(z)$ and $q_{P',H'}(z)$ are $\varepsilon/2K$-close for every $H' \in \mathcal{H}'$. It follows that the functions $\widetilde{t}_{P,H}(z)$ and $\widetilde{t}_{P',H}(z)$ are $\varepsilon/2$-close. Recall that the choice of $\mathcal{H}'$ was independent of P, thus, the functions $t_{P,H}(z)$ and $\widetilde{t}_{P,H}(z)$ are $\varepsilon/4$-close, and the functions $t_{P',H}(z)$ and $\widetilde{t}_{P',H}(z)$ are also $\varepsilon/4$-close. We conclude that the functions $t_{P,H}(z)$ and $t_{P',H}(z)$ are ε-close. □

We are now ready to prove Lemma 11.

Proof of Lemma 11. Suppose that a bounding sequence P, a graph H and a real $\varepsilon > 0$ are given. Apply Lemma 27 with P, H and ε to get a real $\varepsilon_{P,H} > 0$ and an integer $k_{P,H}$, and apply Lemma 22 with $\varepsilon_{P,H}$ to get an integer k_P. We set $k = \max\{k_{P,H}, k_P\}$. Let P' be a bounding sequence that is a k-strengthening of P. Lemma 22 yields that the L_1-distance of P and P' is at most $\varepsilon_{P,H}$. Hence, Lemma 27 implies that the functions $t_{P,H}(z)$ and $t_{P',H}(z)$ are ε-close. □

8 Concluding remarks

Notice that it was not essential in the proof of Theorem 18 that the functions t_k represent densities of G_k in $W_P(z)$. The key property that allowed us to apply the machinery in Section 4 was that the functions t_k are totally analytic functions in a feasible region. While this is not true for all continuous functions on the space of graphons (for example, we need a certain kind of differentiability), many functions besides those corresponding to densities of graphs have this property as well. An example of another function amenable to our methods is the entropy function of a graphon, which is defined for a graphon W as follows:

$$H(W) = \int_{[0,1]^2} W(x,y)\log W(x,y) + (1-W(x,y))\log(1-W(x,y))\,\mathrm{d}x\,\mathrm{d}y\,.$$

The entropy of a graphon W measures the number of graphs close in the cut distance to W [12], and graphons maximizing the entropy among those observing some density conditions correspond to a typical graph satisfying those conditions. There are many classical results on the typical structure of graphs avoiding induced subgraphs, e.g. [1, 18, 26, 27, 39–41], and there is also a good amount of literature, e.g. [11, 36, 42–44], on the typical structure of graphs with general density conditions and the related phase transition phenomenon.

Theorem 18 can be extended to require that every graphon in the family $\mathcal{W}$ has the same non-zero entropy, i.e., the structure of graphs close to graphons in $\mathcal{W}$ is not exponentially concentrated around a single "typical" graphon. More precisely, we obtain the following theorem.

Theorem 28. *There exists a family of graphons $\mathcal{W}$, graphs $H_1, \ldots, H_\ell$ and reals $d_1, \ldots, d_\ell$ such that*

- *a graphon W is weakly isomorphic to a graphon contained in $\mathcal{W}$ if and only if $d(H_i, W) = d_i$ for every $i \in [\ell]$,*
- *all graphons in $\mathcal{W}$ have the same non-zero entropy, and*
- *no graphon in $\mathcal{W}$ is finitely forcible.*

We remark that we cannot use directly the family $\mathcal{W}_P$ of graphons $W_P(z)$ to prove Theorem 28. The reason is that the derivative of the entropy function $H(p) = p\log p + (1-p)\log(1-p)$, which is integrated in the definition of graphon entropy, is not finite for $p \in \{0,1\}$, and the values in some parts of the tile $C \times C$ can approach 0 or 1 as a function of $z \in [0,1]^{\mathbb{N}}$. An easy way to overcome this is to use the family of graphons $\widetilde{W_P}(z)$ defined as $\widetilde{W_P}(z) = W_P(z/2 + 1/4)$. Then the values inside the tile $C \times C$, which depend on $z \in [0,1]^{\mathbb{N}}$, stay bounded away from 0 and 1. Note that this issue does not concern the tile $C \times E$: all values in the tile $C \times E$ are either 0 or 1, which implies that the integral of the entropy function over the tile $C \times E$ is always zero, i.e., the integral is independent of $z \in [0,1]^{\mathbb{N}}$.

The proof of Theorem 18 presented here is not constructive since the bounding sequence P used in the proof is not explicitly constructed. It is worth noting that Cooper et al. in [15], which is an earlier version of [16], showed that any graphon in which the density of all dyadic squares can be approximated by a Turing machine can be embedded in a finitely forcible graphon. While the main result of [15] does not imply that any graphon can be embedded in a finitely forcible graphon, it still provides an alternative way of proving Theorem 18, since the diagonalization arguments can be simulated by a suitably chosen Turing machine. Although this would make the proof of Theorem 18 more technical and significantly longer, we would obtain a proof of Theorem 18 where all the densities $d_1, \dots, d_\ell$ can be taken to be rational, i.e., we would obtain a counterexample to Conjecture 1 involving only constraints that are finite in their nature. In addition, such a proof of Theorem 18 would also allow explicitly constructing the densities $d_1, \dots, d_\ell$.

Finally, we would like to make a short remark about the orders of the graphs $H_1, \dots, H_\ell$ in the statement of Theorem 18. Each decorated constraint that we use involves graphs with at most 15 vertices (including the constraints that are included from the proof Theorem 2 in [16]). Examining the proof of Lemma 5 yields that the orders of graphs get multiplied by the number of parts increased by one, which is 15 in our case. We conclude that all graphs $H_1, \dots, H_\ell$ in the statement of Theorem 18 have at most 225 vertices.

Acknowledgement

The authors would like to thank the anonymous reviewer for carefully reading the submitted manuscript and the comments, which helped to improved the presentation of the results.

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